J. Math. Anal. Appl. 379 (2011) 125–149

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Compactness and invariance properties of evolution operators associated with Kolmogorov operators with unbounded coefficients

Luciana Angiuli [a], Luca Lorenzi [b,*]

[a] Dipartimento di Matematica "Ennio De Giorgi", Università del Salento, Via Per Arnesano, C.P. 193, I-73100 Lecce, Italy
[b] Dipartimento di Matematica, Università di Parma, Parco Area delle Scienze, 53/A, I-43124 Parma, Italy

ARTICLE INFO

Article history:
Received 3 August 2010
Available online 21 December 2010
Submitted by Steven G. Krantz

Keywords:
Nonautonomous second-order elliptic operators
Unbounded coefficients
Evolution operators
Compactness
Invariant subspaces

ABSTRACT

In this paper we consider nonautonomous elliptic operators $\mathcal{A}$ with nontrivial potential term defined in $I \times \mathbb{R}^d$, where I is a right-halfline (possibly $I = \mathbb{R}$). We prove that we can associate an evolution operator $(G(t,s))$ with $\mathcal{A}$ in the space of all bounded and continuous functions on $\mathbb{R}^d$. We also study the compactness properties of the operator $G(t,s)$. Finally, we provide sufficient conditions guaranteeing that each operator $G(t,s)$ preserves the usual L^p-spaces and $C_0(\mathbb{R}^d)$.

1. Introduction

Second-order autonomous elliptic operators with unbounded coefficients have been the subject of many mathematical researches. The interest in such operators comes from their many applications to branches of life sciences such as mathematical finance. Starting from the pioneering papers by Itô [11] and Azencott [3], the literature has spread out considerably and an almost systematic treatment of such operators (and their associated semigroups) is nowadays available. We refer the reader to e.g. [2,4,15] and their rich bibliographies.

On the contrary the study of nonautonomous second-order elliptic operators is at a preliminary level. The pioneering paper is [5] where the nonautonomous Ornstein–Uhlenbeck operator

$$\big(\mathcal{L}(t)\psi\big)(x) = \sum_{i,j=1}^{d} q_{ij}(t)D_{ij}\psi(x) + \sum_{i,j=1}^{d} b_{ij}(t)x_j D_i\psi(x), \quad (t,x) \in \mathbb{R}^{1+d},$$

has been studied in the case when its coefficients are T-periodic for some $T > 0$. The analysis of [5] has been continued in a couple of papers by Geissert and Lunardi (see [9,10]) where $\mathcal{L}$ and the associated evolution operator $(L(t,s))$ have been extensively studied both in periodic and nonperiodic settings.

Recently, in [12] the more general nonautonomous elliptic operator

* Corresponding author.
E-mail addresses: luciana.angiuli@unisalento.it (L. Angiuli), luca.lorenzi@unipr.it (L. Lorenzi).
URL: http://www.unipr.it/~lorluc99/index.html (L. Lorenzi).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.12.029

$$\big(\mathcal{A}(t)\psi\big)(x)=\sum_{i,j=1}^d q_{ij}(t,x)D_{ij}\psi(x)+\sum_{j=1}^d b_j(t,x)D_j\psi(x),\quad (t,x)\in I\times\mathbb{R}^d,$$

has been studied, when I is a right-halfline (possibly $I=\mathbb{R}$). Under rather mild regularity conditions on its coefficients and assuming the ellipticity condition

$$\sum_{i,j=1}^d q_{ij}(t,x)\xi_i\xi_j\geqslant\eta_0|\xi|^2,\quad (t,x)\in I\times\mathbb{R}^d,\ \xi\in\mathbb{R}^d,$$

for some positive constant η_0, the existence of a (unique) evolution operator $(G(t,s))$ associated with $\mathcal{A}$ in $C_b(\mathbb{R}^d)$ (the space of all bounded and continuous functions $f:\mathbb{R}^d\to\mathbb{R}$) has been proved. The main properties of the evolution operator $(G(t,s))$ in $C_b(\mathbb{R}^d)$ have been extensively studied and the authors extended many of the results proved for the Ornstein–Uhlenbeck operator.

In this paper, we are interested in studying nonautonomous elliptic operators in $C_b(\mathbb{R}^d)$, with a nonzero potential term, i.e., we are interested in operators of the form

$$\big(\mathcal{A}(t)\psi\big)(x)=\sum_{i,j=1}^d q_{ij}(t,x)D_{ij}\psi(x)+\sum_{j=1}^d b_j(t,x)D_j\psi(x)-c(t,x)\psi(x),$$

for any $(t,x)\in I\times\mathbb{R}^d$, where c is bounded from below and I is as above. Adapting the arguments used in the case of no potential term, we first show in Section 2 that we can associate an evolution operator $(G(t,s))$ in $C_b(\mathbb{R}^d)$ with the operator $\mathcal{A}$. In fact, $G(t,s)f$ can be obtained as the "limit" as $n\to+\infty$, in an appropriate sense, of both the sequences of solutions to the Cauchy–Dirichlet and Cauchy–Neumann problems for the equation $D_tu-\mathcal{A}u=0$ in the ball $B(0,n)$. Next, in Section 3 we show that it is possible to associate a Green function g with the evolution operator $(G(t,s))$, namely,

$$\big(G(t,s)f\big)(x)=\int_{\mathbb{R}^d} g(t,s,x,y)f(y)\,dy,\quad s,t\in I,\ s<t,\ x\in\mathbb{R}^d,\tag{1.1}$$

for every $f\in C_b(\mathbb{R}^d)$. For any fixed s and almost any $y\in\mathbb{R}^d$, $g(\cdot,s,\cdot,y)$ is smooth and solves the equation $D_tg-\mathcal{A}g=0$. Formula (1.1) allows us to extend each operator $G(t,s)$ to the space $B_b(\mathbb{R}^d)$ of all bounded and Borel measurable functions $f:\mathbb{R}^d\to\mathbb{R}$. The so extended operators turn out to be strong Feller (i.e., $G(t,s)$ maps $B_b(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$) and irreducible (i.e., if $U\neq\emptyset$ is a Borel measurable set, then $(G(t,s)\chi_U)(x)>0$ for any $x\in\mathbb{R}^d$ and any $s<t$). We also prove that, for any continuous function $f:\mathbb{R}^d\to\mathbb{R}$, vanishing at infinity, and any $t\in I$, the function $G(t,\cdot)f$ is continuous in $(-\infty,t]$ with values in $C_b(\mathbb{R}^d)$, for any $t\in I$. We then deduce that $G(\cdot,\cdot)f$ is continuous in $\{(t,s,x)\in I\times I\times\mathbb{R}^d\colon t\geqslant s\}$. Finally, under an additional assumption, we establish an integral inequality which will play a crucial role in what follows.

Section 4 is devoted to the study of the compactness of the operator $G(t,s)$ in $C_b(\mathbb{R}^d)$, thus extending a similar result recently proved in [14] in the case when the potential c identically vanishes. We stress that, already in the classical case when the coefficients of the operator $\mathcal{A}$ are bounded, the operator $G(t,s)$ is, in general, not compact (see Remark 4.6). Hence, additional assumptions on the coefficients of the operator $\mathcal{A}$ should be assumed to prove that $G(t,s)$ is compact. In fact, given an interval $J\subset I$ we show that $G(t,s)$ is compact for any $s,t\in J$, such that $s<t$, if and only if the family of measures $\{g(t,s,x,y)\,dy,\ x\in\mathbb{R}^d\}$ (which are not probability measures if $c\not\equiv 0$) is tight for any s,t as above, where tightness means that for any $\varepsilon>0$ there exists $R_0>0$ such that

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d\setminus B(0,R)} g(t,s,x,y)\,dy\leqslant\varepsilon,$$

provided that $R\geqslant R_0$. A sufficient condition is then provided for the previous family of measures be tight and consequently we show that, in this case, $G(t,s)$ preserves neither $C_0(\mathbb{R}^d)$ nor $L^p(\mathbb{R}^d)$. Such condition turns out to be stronger than that of the case when $c\equiv 0$. Indeed, one of the main tools of the proof is Proposition 4.3 which is trivially satisfied in the case when $c\equiv 0$ and the Cauchy problem $D_tu(t,\cdot)=\mathcal{A}(t)u(t,\cdot)$ $(t>s)$, $u(s,\cdot)=\mathbb{1}$ is uniquely solvable, since in this case $G(t,s)\mathbb{1}=\mathbb{1}$. Another main tool in the proof of the compactness of the evolution operator is the formula (3.6) which, in the case when $c\equiv 0$, is an equality and it can be proved rather easily. The presence of the potential term leads to some additional technical difficulties.

Adapting some of the ideas in the proof of Theorem 4.4, we then provide a sufficient condition to guarantee that the function $G(\cdot,\cdot)f$ is continuous in $\{(t,s,x)\in I\times I\times\mathbb{R}^d\colon t\geqslant s\}$ for any bounded and continuous function $f:\mathbb{R}^d\to\mathbb{R}$, thus extending the similar result of Section 3 proved for functions vanishing at infinity.

It is well known that, when a family of probability measures $\{\mu_t\colon t\in I\}$ with the property

$$\int_{\mathbb{R}^d} G(t,s)f\,d\mu_t=\int_{\mathbb{R}^d} f\,d\mu_s,\quad s<t,\ f\in C_b\big(\mathbb{R}^d\big),$$

exists, each operator $G(t,s)$ extends to a bounded linear operator from $L^p(\mathbb{R}^d,\mu_s)$ to $L^p(\mathbb{R}^d,\mu_t)$ for any $p\in[1,+\infty)$. The family $\{\mu_t\colon t\in I\}$, which is called *evolution system of invariant measures* in [6] and *entrance laws at* $-\infty$ in [7], consists of measures which are equivalent to the Lebesgue measure. As a matter of fact, the measures μ_t are not known explicitly, in general, and the L^p-spaces related to two different measures μ_t and μ_s differ. Also in view of these difficulties, it is interesting to determine sufficient conditions which guarantee that the usual L^p-spaces related to the Lebesgue measure are preserved under the action of the operator $G(t,s)$.

Under our assumptions, whenever $G(t,s)$ is compact it does not preserve $C_0(\mathbb{R}^d)$. It thus makes sense to look for sufficient conditions for $C_0(\mathbb{R}^d)$ be preserved by $G(t,s)$. The study of the invariance of $C_0(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$ ($p\in[1,+\infty)$), under the action of $G(t,s)$, are the contents of Section 5. In particular, under the assumptions that guarantee that $C_0(\mathbb{R}^d)$ is invariant under the action of the evolution operator $(G(t,s))$, we show that the restriction of $(G(t,s))$ to $C_0(\mathbb{R}^d)$ gives rise to a strongly continuous evolution operator.

Finally, in Section 6 examples of nonautonomous operators to which the main results of this paper apply are provided.

Notations

We denote by $B_b(\mathbb{R}^d)$ the Banach space of all bounded and Borel measurable functions $f:\mathbb{R}^d\to\mathbb{R}$, and by $C_b(\mathbb{R}^d)$ its subspace of all continuous functions. $B_b(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$ are endowed with the sup norm $\|\cdot\|_\infty$. For $k>0$ $C_b^k(\mathbb{R}^d)$ is the set of all functions $f\in C_b(\mathbb{R}^d)$ whose derivatives up to the $[k]$th-order are bounded and $(k-[k])$-Hölder continuous in $\mathbb{R}^d$. Here, $[k]$ denotes the integer part of k. We use the subscript "c" (resp. "0") instead of "b" for spaces of functions with compact support (resp. for spaces of functions vanishing at infinity).

Let $\mathcal{O}\subset\mathbb{R}^{1+d}$ be an open set or the closure of an open set. For $0<\alpha<1$ we denote by $C^{\alpha/2,\alpha}_{\rm loc}(\mathcal{O})$ the set of functions $f:\mathcal{O}\to\mathbb{R}$ whose restrictions to any compact set $\mathcal{O}_0\subset\mathcal{O}$ belong to $C^{\alpha/2,\alpha}(\mathcal{O}_0)$. Similarly, $C^{1+\alpha/2,2+\alpha}_{\rm loc}(\mathcal{O})$ is the subset of $C(\mathcal{O})$ of the functions f such that the time derivative $D_tf:=\frac{\partial f}{\partial t}$ and the spatial derivatives $D_if:=\frac{\partial f}{\partial x_i}$, $D_{ij}f:=\frac{\partial^2 f}{\partial x_i\partial x_j}$ exist and belong to $C^{\alpha/2,\alpha}_{\rm loc}(\mathcal{O})$.

We denote by $\mathrm{Tr}(Q)$ and $\langle x,y\rangle$ the trace of the square matrix Q and the Euclidean scalar product of the vectors $x,y\in\mathbb{R}^d$, respectively. By χ_A we denote the characteristic function of the set $A\subset\mathbb{R}^d$ and by $\mathbb{1}$ we denote the function which is identically equal to 1 in $\mathbb{R}^d$.

For any interval $I\subset\mathbb{R}$, we set $\Lambda_I:=\{(t,s)\in I\times I\colon t\geqslant s\}$. For any $t\in I$, we denote by I_t the intersection of I and $(-\infty,t]$. Further, for any real-valued function ξ, defined in the same set where the coefficients of the operator $\mathcal{A}$ are defined, we denote by $\mathcal{A}_\xi$ the operator $\mathcal{A}-\xi I$. Finally, by $a\vee b$ and $a\wedge b$ we denote, respectively, the maximum and the minimum between $a,b\in\mathbb{R}$.

2. The evolution operator

Let I be an interval, which is either $\mathbb{R}$ or a right-halfline, and let the operators $\mathcal{A}(t)$, $t\in I$, be defined on smooth functions ψ by

$$\big(\mathcal{A}(t)\psi\big)(x)=\sum_{i,j=1}^d q_{ij}(t,x)D_{ij}\psi(x)+\sum_{i=1}^d b_i(t,x)D_i\psi(x)-c(t,x)\psi(x),$$

for any $(t,x)\in I\times\mathbb{R}^d$, under the following hypothesis:

Hypothesis 2.1.

(i) q_{ij}, b_i $(i,j=1,\dots,d)$ and c belong to $C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d)$;
(ii) $c_0:=\inf_{I\times\mathbb{R}^d}c>-\infty$;
(iii) for every $(t,x)\in I\times\mathbb{R}^d$, the matrix $Q(t,x)=(q_{ij}(t,x))$ is symmetric and there exists a function $\eta:I\times\mathbb{R}^d\to\mathbb{R}$ such that $0<\eta_0:=\inf_{I\times\mathbb{R}^d}\eta$ and

$$\big\langle Q(t,x)\xi,\xi\big\rangle\geqslant\eta(t,x)|\xi|^2,\quad \xi\in\mathbb{R}^d,\ (t,x)\in I\times\mathbb{R}^d;$$

(iv) for every bounded interval $J\subset I$ there exist a positive function $\varphi=\varphi_J\in C^2(\mathbb{R}^d)$ and a real number $\lambda=\lambda_J$ such that

$$\lim_{|x|\to+\infty}\varphi(x)=+\infty\quad\text{and}\quad\big(\mathcal{A}(t)\varphi\big)(x)-\lambda\varphi(x)\leqslant 0,\quad (t,x)\in J\times\mathbb{R}^d.$$

We start by proving a maximum principle.

Proposition 2.2. *Let $s \in I$, $T > s$ and $R > 0$. If $u \in C_b([s,T] \times \mathbb{R}^d \setminus B(0,R)) \cap C^{1,2}((s,T] \times \mathbb{R}^d \setminus \overline{B(0,R)})$ satisfies*

$$\begin{cases} D_t u(t,x) - \mathcal{A}(t)u(t,x) \leqslant 0, & (t,x) \in (s,T] \times \mathbb{R}^d \setminus \overline{B(0,R)}, \\ u(t,x) \leqslant 0, & (t,x) \in [s,T] \times \partial B(0,R), \\ u(s,x) \leqslant 0, & x \in \mathbb{R}^d \setminus B(0,R), \end{cases}$$

then $u \leqslant 0$. Similarly, if $u \in C_b([s,T] \times \mathbb{R}^d) \cap C^{1,2}((s,T] \times \mathbb{R}^d)$ satisfies

$$\begin{cases} D_t u(t,x) - \mathcal{A}(t)u(t,x) \leqslant 0, & (t,x) \in (s,T] \times \mathbb{R}^d, \\ u(s,x) \leqslant 0, & x \in \mathbb{R}^d, \end{cases}$$

then $u \leqslant 0$. In particular, if $u \in C_b([s,T] \times \mathbb{R}^d) \cap C^{1,2}((s,T] \times \mathbb{R}^d)$ solves the Cauchy problem

$$\begin{cases} D_t u(t,x) - \mathcal{A}(t)u(t,x) = 0, & (t,x) \in (s,T] \times \mathbb{R}^d, \\ u(s,x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

then

$$\left\| u(t,\cdot) \right\|_\infty \leqslant e^{-c_0(t-s)} \|f\|_\infty, \quad t > s.$$

Proof. The proof is similar to that of the autonomous case. For the reader's convenience we go into details.

Let $\lambda := \lambda_{[s,T]}$ and $\varphi := \varphi_{[s,T]}$. Without loss of generality we can assume that $\lambda > -c_0$. As it is immediately seen, for any $n \in \mathbb{N}$, the function $v_n(t,x) = e^{-\lambda(t-s)}u(t,x) - n^{-1}\varphi(x)$ satisfies the inequalities

$$\begin{cases} D_t v_n(t,x) - \mathcal{A}_\lambda(t)v_n(t,x) \leqslant 0, & (t,x) \in (s,T] \times \mathbb{R}^d \setminus \overline{B(0,R)}, \\ v_n(t,x) \leqslant 0, & (t,x) \in [s,T] \times \partial B(0,R), \\ v_n(s,x) \leqslant 0, & x \in \mathbb{R}^d \setminus B(0,R). \end{cases}$$

Since u is bounded in $[s,T] \times \mathbb{R}^d$ and φ blows up as $|x| \to +\infty$, the function v_n tends to $-\infty$ as $|x| \to +\infty$, uniformly with respect to $t \in [s,T]$. Hence, it has a maximum at some point (t_0,x_0). Such a maximum cannot be positive, otherwise it would be $t_0 > s$ and $x_0 \in \mathbb{R}^d \setminus \overline{B(0,R)}$, and from the differential inequality we would be led to a contradiction. Hence, $v_n \leqslant 0$ in $[s,T] \times \mathbb{R}^d \setminus B(0,R)$. Letting $n \to +\infty$, yields $u \leqslant 0$ in $[s,T] \times \mathbb{R}^d \setminus B(0,R)$. Clearly, the same proof can be applied to show the second statement of the theorem.

To prove the last part of the statement, it suffices to consider the functions $v_\pm$ defined by $v_\pm(t,x) = \pm e^{c_0(t-s)}u(t,x) - \|f\|_\infty$ for any $(t,x) \in [s,T] \times \mathbb{R}^d$, which satisfy the differential inequalities

$$\begin{cases} D_t v_\pm(t,x) - \mathcal{A}_{-c_0}(t)v_\pm(t,x) \leqslant 0, & (t,x) \in (s,T] \times \mathbb{R}^d, \\ v_\pm(s,x) \leqslant 0, & x \in \mathbb{R}^d. \end{cases}$$

The previous results, applied to the operator $\mathcal{A}_{-c_0}$ (which clearly satisfies Hypothesis 2.1), show that $v_\pm(t,x) \leqslant 0$ for any $(t,x) \in [s,T] \times \mathbb{R}^d$ and this gives the assertion at once. $\square$

We can now prove an existence-uniqueness result for the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}(t)u(t,x), & (t,x) \in (s,+\infty) \times \mathbb{R}^d, \\ u(s,x) = f(x), & x \in \mathbb{R}^d, \end{cases} \tag{2.1}$$

with datum $f \in C_b(\mathbb{R}^d)$. For this purpose for any $n \in \mathbb{N}$ we introduce the Cauchy problems

$$\begin{cases} D_t u_n(t,x) = \mathcal{A}(t)u_n(t,x), & (t,x) \in (s,+\infty) \times B(0,n), \\ u_n(t,x) = 0, & (t,x) \in (s,+\infty) \times \partial B(0,n), \\ u_n(s,x) = f(x), & x \in B(0,n) \end{cases} \tag{2.2}$$

and

$$\begin{cases} D_t u_n(t,x) = \mathcal{A}(t)u_n(t,x), & (t,x) \in (s,+\infty) \times B(0,n), \\ \dfrac{\partial u_n}{\partial \nu}(t,x) = 0, & (t,x) \in (s,+\infty) \times \partial B(0,n), \\ u_n(s,x) = f(x), & x \in B(0,n), \end{cases} \tag{2.3}$$

where $\nu = \nu(x)$ denotes the exterior unit normal at $x \in \partial B(0,n)$. We further denote by $G_n^D(\cdot,s)$ and $G_n^N(\cdot,s)$ the bounded operators on $C_b(\mathbb{R}^d)$ which associate with any $f \in C_b(\mathbb{R}^d)$ the unique classical solution to problems (2.2) and (2.3), respectively.

Theorem 2.3. *For any $f \in C_b(\mathbb{R}^d)$ and any $s \in I$ the Cauchy problem (2.1) admits a unique solution $u_f \in C([s,+\infty)\times\mathbb{R}^d) \cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ (α being given by Hypothesis 2.1(i)), which is bounded in $[s,T]\times\mathbb{R}^d$ for any $T>s$. For any $t>s$ and any $f\in C_b(\mathbb{R}^d)$, set $G(t,s)f := u_f(t,\cdot)$. Then, $G(t,s)$ is a bounded linear operator in $C_b(\mathbb{R}^d)$ and*

$$\|G(t,s)\|_{\mathcal{L}(C_b(\mathbb{R}^d))} \leqslant e^{-c_0(t-s)}, \quad t \geqslant s. \tag{2.4}$$

Moreover, the following properties hold true:

(i) *for any $f \in C_b(\mathbb{R}^d)$, $G_n^N(\cdot,s)f$ converges to $G(\cdot,s)f$ in $C^{1,2}(D)$ for any compact set $D\subset(s,+\infty)\times\mathbb{R}^d$;*
(ii) *for any $f\in C_b(\mathbb{R}^d)$ and any $s\in I$, the function $G_n^D(\cdot,s)f$ converges to $G(\cdot,s)f$ in $C^{1,2}(D)$ for any compact set $D\subset(s,+\infty)\times\mathbb{R}^d$. Moreover, if f is nonnegative, then $G_n(t,s)f$ is increasing to $G(t,s)f$ for any $(t,s)\in\Lambda_I$.*

Proof. Let us prove the first part of the statement and property (i). The uniqueness of the solution to problem (2.1) and estimate (2.4) follow from Proposition 2.2. Let us now prove that, for any $f\in C_b(\mathbb{R}^d)$, $G_n^N(\cdot,s)f$ converges, up to a subsequence, to a solution to problem (2.1) which satisfies the properties in the statement of the theorem. For this purpose we fix $f\in C_b(\mathbb{R}^d)$. The Schauder estimates in [13, Theorems IV.5.3, IV.10.1] show that the sequence $\|G_n^N(\cdot,s)f\|_{C^{1+\alpha/2,2+\alpha}(K)}$ is bounded, for any compact set $K\subset(s,T)\times\mathbb{R}^d$, by a constant independent of n. The Arzelà–Ascoli theorem, the arbitrariness of K and a diagonal argument allow to conclude that there exists a subsequence $(G_{n_k}^N(\cdot,s)f)$ which converges to a function $u\in C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ in $C^{1,2}(D)$, for any compact set $D\subset(s,+\infty)\times\mathbb{R}^d$. Clearly, u satisfies the differential equation in (2.1). Hence, to prove that u solves problem (2.1) we just need to show that u is continuous at $t=s$ and it therein equals the function f. As a byproduct we also then deduce that the whole sequence $(G_n^N(\cdot,s)f)$ converges to u in $C^{1,2}(D)$ for any compact set $D\subset(s,+\infty)\times\mathbb{R}^d$, since our arguments show that any subsequence of $(G_n^N(\cdot,s)f)$ has a subsequence converging to u in $C^{1,2}(D)$ for any D as above.

Let us first suppose that $f\in C_c^{2+\alpha}(\mathbb{R}^d)$. In this case we can estimate $\|G_n^N(\cdot,s)f\|_{C^{1+\alpha/2,2+\alpha}(D)}$ from above by a constant, which is independent of n, for any compact set $D\subset[s,+\infty)\times\mathbb{R}^d$ and any $n\in\mathbb{N}$ such that $\mathrm{supp}(f)\subset B(0,n)$. Hence, $G_{n_k}^N(\cdot,s)f$ converges to u uniformly in D and, as a byproduct, u is continuous up to $t=s$ and is a solution to problem (2.1).

Let us now assume that $f\in C_0(\mathbb{R}^d)$ and let $(f_m)\subset C_c^{2+\alpha}(\mathbb{R}^d)$ converge to f uniformly in $\mathbb{R}^d$. Then, using the classical maximum principle, which shows that $\|G_n^N(t,s)g\|_\infty \leqslant e^{-c_0(t-s)}\|g\|_\infty$ for any $g\in C(\overline{B(0,n)})$ and any $n\in\mathbb{N}$, we can estimate

$$\begin{aligned}\big|\big(G_{n_k}^N(t,s)f\big)(x)-f(x)\big| &\leqslant \big|\big(G_{n_k}^N(t,s)f\big)(x)-\big(G_{n_k}^N(t,s)f_m\big)(x)\big|\\ &\quad+\big|\big(G_{n_k}^N(t,s)f_m\big)(x)-f_m(x)\big|+\big|f_m(x)-f(x)\big|\\ &\leqslant \big(e^{-c_0(t-s)}+1\big)\|f-f_m\|_\infty+\big|\big(G_{n_k}^N(t,s)f_m\big)(x)-f_m(x)\big|,\end{aligned}$$

for any $t>s$ and any $x\in\mathbb{R}^d$. Letting $k\to+\infty$ yields

$$\big|u(t,x)-f(x)\big|\leqslant\big(e^{-c_0(t-s)}+1\big)\|f-f_m\|_\infty+\big|u_{f_m}(t,x)-f_m(x)\big|,$$

for any $(t,x)\in(s,+\infty)\times\mathbb{R}^d$ and any $m\in\mathbb{N}$, which clearly implies that $u(t,\cdot)$ tends to f as $t\to s^+$, locally uniformly in $\mathbb{R}^d$.

To conclude, let us consider the case when f is merely bounded and continuous in $\mathbb{R}^d$. Fix $R>0$ and let $\eta\in C_c^{2+\alpha}(\mathbb{R}^d)$ satisfy $\eta\equiv1$ in $B(0,R)$ and $0\leqslant\eta\leqslant1$ in $\mathbb{R}^d$. Further, let $(f_n)\subset C_c^{2+\alpha}(\mathbb{R}^d)$ be a bounded sequence with respect to the sup-norm converging to f locally uniformly in $\mathbb{R}^d$, and set $M=\sup_{n\in\mathbb{N}}\|f_n\|_\infty$. Note that

$$\big|G_n^N(t,s)\big((\mathbb{1}-\eta)f_n\big)\big|\leqslant\|f_n\|_\infty G_n^N(t,s)(\mathbb{1}-\eta)\leqslant M\big(e^{-c_0(t-s)}-G_n^N(t,s)\eta\big),$$

for any $s<t$ and any $n\in\mathbb{N}$, as it follows immediately from the positivity of each operator $G_n^N(t,s)$. Hence, we can estimate

$$\begin{aligned}\big|\big(G_{n_k}^N(t,s)f\big)(x)-f(x)\big| &\leqslant\big|\big(G_{n_k}^N(t,s)(f-f_n)\big)(x)\big|+\big|\big(G_{n_k}^N(t,s)\big(f_n(\mathbb{1}-\eta)\big)\big)(x)\big|\\ &\quad+\big|\big(G_{n_k}^N(t,s)(f_n\eta)\big)(x)-(f_n\eta)(x)\big|+\big|f_n(x)-f(x)\big|\\ &\leqslant e^{-c_0(t-s)}\|f-f_n\|_{L^\infty(B(0,n_k))}+M\big(e^{-c_0(t-s)}-\big(G_{n_k}^N(t,s)\eta\big)(x)\big)\\ &\quad+\big|\big(G_{n_k}^N(t,s)(f_n\eta)\big)(x)-(f_n\eta)(x)\big|+\big|f_n(x)-f(x)\big|,\end{aligned}$$

for any $x\in B(0,R)$ and any k such that $n_k>R$. Letting first $n\to+\infty$ and then $k\to+\infty$ we get

$$\big|u(t,x)-f(x)\big|\leqslant M\big(e^{-c_0(t-s)}-u_\eta(t,x)\big)+\big|u_{f\eta}(t,x)-(f\eta)(x)\big|,$$

for any $x\in B(0,R)$. Letting $t\to s^+$, we see that $u(t,\cdot)\to f$, uniformly in $B(0,R)$.

The proof of property (ii) follows the same lines of the proof of property (i). Hence, we skip the details. We just observe that the pointwise convergence of the sequence $(G_n^D(t,s)f)$ can also be proved applying the classical maximum principle to the function $G_m^D(\cdot,s)f-G_n^D(\cdot,s)f$ ($n,m\in\mathbb{N}$, $m>n$), which shows that, if $f\geqslant0$, then $G_m^D(\cdot,s)f-G_n^D(\cdot,s)f\geqslant0$ in $[s,+\infty)\times B(0,n)$. $\square$

3. Basic properties of the operators $G(t,s)$

Let us now prove some properties of the operators $G(t,s)$. For this purpose, we set $G(t,t) := id_{C_b(\mathbb{R}^d)}$.

Proposition 3.1 *(Green kernel). The following properties are satisfied.*

(i) *The family of operators $G(t,s)$ $(t,s \in I,\ s<t)$ defines an evolution operator on $C_b(\mathbb{R}^d)$, i.e., each operator $G(t,s)$ is bounded from $C_b(\mathbb{R}^d)$ into itself, $G(s,s)$ is the identity operator, and $G(t,s)G(s,r) = G(t,r)$ for any $I \ni r < s < t$.*

(ii) *The evolution operator $(G(t,s))$ can be represented in the form*

$$\big(G(t,s)f\big)(x) = \int_{\mathbb{R}^d} g(t,s,x,y) f(y)\,dy, \quad s<t,\ x \in \mathbb{R}^d, \tag{3.1}$$

for any $f \in C_b(\mathbb{R}^d)$, where $g : \Lambda_I \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a positive function. For any $s \in I$ and almost any $y \in \mathbb{R}^d$, $g(\cdot,s,\cdot,y)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ and solves the equation $D_t g - \mathcal{A}g = 0$ in $(s,+\infty)\times\mathbb{R}^d$. Moreover,

$$\big\|g(t,s,x,\cdot)\big\|_{L^1(\mathbb{R}^d)} \leqslant e^{-c_0(t-s)}, \quad s<t,\ x \in \mathbb{R}^d. \tag{3.2}$$

The function g is called the Green function of $D_t u - \mathcal{A}u = 0$ in $(s,+\infty)\times\mathbb{R}^d$.

(iii) *$G(t,s)$ can be extended to $B_b(\mathbb{R}^d)$ through formula (3.1). Each operator $G(t,s)$ is irreducible and has the strong Feller property.*

Proof. (i) It follows from the uniqueness of the solution to problem (2.1). Indeed, for any $r<s$ and any $f \in C_b(\mathbb{R}^d)$, the function $G(\cdot,r)f$ belongs to $C([s,+\infty)\times\mathbb{R}^d) \cap C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$, is bounded in $[s,T]\times\mathbb{R}^d$ for any $T>s$ and solves the Cauchy problem

$$\begin{cases} D_t u(t,x) = \mathcal{A}(t)u(t,x), & (t,x) \in (s,+\infty)\times\mathbb{R}^d,\\ u(s,x) = \big(G(s,r)f\big)(x), & x \in \mathbb{R}^d. \end{cases}$$

Hence, by uniqueness, $G(t,r)f = G(t,s)G(s,r)f$ for any $t>s$.

(ii) By [8, Theorem 3.7.16] we know that for every $n \in \mathbb{N}$ there exists a unique Green function g_n of the Cauchy–Dirichlet problem (2.2) in $(s,+\infty)\times B(0,n)$, i.e., a unique function g_n such that

$$\big(G_n^D(t,s)f\big)(x) = \int_{B(0,n)} g_n(t,s,x,y) f(y)\,dy, \quad t>s,\ x \in B(0,n),$$

for any $f \in C(\overline{B(0,n)})$. The function g_n is positive and, as a function of (t,x), it belongs to $C^{1+\alpha/2,2+\alpha}((\tau,T)\times B(0,n))$ for every fixed $y \in B(0,n)$, $s \in I$ and $s<\tau<T$. Moreover, it satisfies $D_t g_n - \mathcal{A}g_n = 0$ in $(s,+\infty)\times B(0,n)$. By Theorem 2.3(ii), for any nonnegative $f \in C_b(\mathbb{R}^d)$, the sequence $((G_n^D(t,s)f)(x))$ increases to $(G(t,s)f)(x)$. As a byproduct, the functions g_n increase with n. Therefore, defining

$$g(t,s,x,y) = \lim_{n\to+\infty} g_n(t,s,x,y), \quad (t,s,x,y) \in \Lambda_I \times \mathbb{R}^d \times \mathbb{R}^d,$$

by monotone convergence we get that

$$\big(G(t,s)f\big)(x) = \lim_{n\to+\infty} \big(G_n^D(t,s)f\big)(x) = \int_{\mathbb{R}^d} g(t,s,x,y) f(y)\,dy,$$

for any $f \geqslant 0$. For a general $f \in C_b(\mathbb{R}^d)$ it suffices to split $f = f^+ - f^-$ and to apply the above argument to f^+ and f^-. This shows that (3.1) holds. The positivity of g is obvious since each function g_n is positive in $\Lambda_I \times B(0,n) \times B(0,n)$. By (2.4) we have that

$$\int_{\mathbb{R}^d} g(t,s,x,y)\,dy = \big(G(t,s)\mathbb{1}\big)(x) \leqslant e^{-c_0(t-s)}, \quad t \geqslant s,\ x \in \mathbb{R}^d,$$

and (3.2) is proved.

As far as the regularity of g with respect to the variables t,x is concerned, we first show that, for every $s \in I$ and almost all $y \in \mathbb{R}^d$, $g_n(\cdot,s,\cdot,y)$ is locally bounded in $I \times \mathbb{R}^d$, uniformly with respect to n. Once this property is checked, the same argument used in the proof of Theorem 2.3(i), based on interior Schauder estimates and the Arzelà–Ascoli theorem, will show that $g(\cdot,s,\cdot,y)$ belongs to $C^{1+\alpha/2,2+\alpha}_{\rm loc}((s,+\infty)\times\mathbb{R}^d)$ for every $s \in I$ and almost any $y \in \mathbb{R}^d$. So, let us fix two compact sets $[\tau,T] \subset (s,+\infty)$ and $K \subset \mathbb{R}^d$. Further, denote by (t_h) and (x_k) two countable sets dense in $[\tau,T+1]$ and in K, respectively. Since $\int_{\mathbb{R}^d} g(t_h,s,x_k,y)\,dy < +\infty$ for any $h,k \in \mathbb{N}$, there exists a set $\mathcal{Y} \subset \mathbb{R}^d$ with negligible complement such that $g(t_h, s, x_k, y) < +\infty$ for any $y \in \mathcal{Y}$ and any $h, k \in \mathbb{N}$. Let $\bar{y} \in \mathcal{Y}$ and let R be sufficiently large such that $s < \tau - 2/R$ and $\bigcup_{x \in K} \overline{B(x, 1)} \subset B(0, R)$. Moreover, let ϑ be a smooth function compactly supported in $[\tau - 2/R, T + 2] \times \overline{B(0, R+1)}$ such that $0 \leqslant \vartheta \leqslant 1$ and $\vartheta(t, x) = 1$ for any $(t, x) \in [\tau - 1/R, T + 1] \times B(0, R)$. Define the operator $\tilde{\mathcal{A}}$ by setting

$$\big(\tilde{\mathcal{A}}(t)\psi\big)(x) = \sum_{i,j=1}^{d} \tilde{q}_{ij}(t, x) D_{ij}\psi(x) + \sum_{j=1}^{d} \tilde{b}_j(t, x) D_j\psi(x) - \tilde{c}(t, x)\psi(x),$$

where

$$\tilde{q}_{ij}(t, x) = \vartheta(t, x) q_{ij}(t, x) + \big(1 - \vartheta(t, x)\big)\delta_{ij},$$
$$\tilde{b}_j(t, x) = \vartheta(t, x) b_j(t, x),$$
$$\tilde{c}(t, x) = \vartheta(t, x) c(t, x),$$

for any $(t, x) \in \mathbb{R}^{1+d}$ and any $i, j = 1, \ldots, d$. Since the function $g_n(\cdot, s, \cdot, y)$ satisfies the equation $D_t g_n(\cdot, s, \cdot, y) - \tilde{\mathcal{A}} g_n(\cdot, s, \cdot, y) = 0$ in $[\tau - 1/R, T + 1] \times B(0, R)$, for any $n > R$, applying the Harnack inequality in [17, Theorem 1], we see that, if $\rho^2 < 1 \wedge 1/R$, then there exists a positive constant M_0, independent of h, k and n, such that

$$g_n(t, s, x, \bar{y}) \leqslant M_0 g_n(t_h, s, x_k, \bar{y}) \leqslant M_0 g(t_h, s, x_k, \bar{y}), \tag{3.3}$$

for every $t \in [t_h - \frac{3}{4}\rho^2, t_h - \frac{1}{2}\rho^2]$ and every $x \in \overline{B(x_k, \rho/2)}$. Since $[\tau, T] \times K$ can be covered by a finite number of cylinders $[t_h - \frac{3}{4}\rho^2, t_h - \frac{1}{2}\rho^2] \times B(x_k, \rho/2)$, from (3.3) we deduce that $g_n(\cdot, s, \cdot, \bar{y})$ is uniformly bounded in $[\tau, T] \times K$ by a constant independent of n, as it has been claimed.

(iii) Clearly, the operator $G(t, s)$ can be extended to the set of all bounded Borel measurable functions f through formula (3.1), which, in its turn, also shows that $G(t, s)$ is irreducible. To prove that $G(t, s)$ is strong Feller, we have to show that, for any $f \in B_b(\mathbb{R}^d)$, $G(t, s)f$ is continuous. In fact, we will show that $G(t, s)f \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$. For this purpose, we fix a bounded sequence (f_n) of bounded and continuous functions converging pointwise to f, almost everywhere in $\mathbb{R}^d$, as $n \to +\infty$. Clearly, $G(t, s)f_n$ converges to $G(t, s)f$ pointwise in $\mathbb{R}^d$ by dominated convergence. Using the Schauder interior estimates, one can easily deduce that, for any $R > 0$, the sequence $(G(t, s)f_n)$ is bounded in $C^{2+\alpha}(B(0, R))$. Hence, the Arzelà–Ascoli theorem implies that $G(t, s)f_n$ converges in $C^2(B(0, R))$ to $G(t, s)f$ and $G(t, s)f$ belongs to $C^{2+\alpha}(B(0, R))$. This completes the proof. $\square$

The following corollary is an immediate consequence of Proposition 3.1. Hence, we skip the proof.

Corollary 3.2. *For every $(t, s) \in \Lambda_I$ and every $x \in \mathbb{R}^d$ let us define the measure $g_{t,s}(x, dy)$ by setting $g_{t,t}(x, dy) = \delta_x$ and*

$$g_{t,s}(x, A) = \int_A g(t, s, x, y)\, dy, \quad t > s, \tag{3.4}$$

for any Borel set $A \subset \mathbb{R}^d$. Then, each measure $g_{t,s}(x, dy)$ is equivalent to the Lebesgue measure (i.e., it has the same sets with zero measure as the restriction of the Lebesgue measure to the σ-algebra of all the Borel sets of $\mathbb{R}^d$). Moreover, for any $t \geqslant r \geqslant s$, $x \in \mathbb{R}^d$ and any Borel set $A \subset \mathbb{R}^d$ it holds that

$$g_{t,s}(x, A) = \int_{\mathbb{R}^d} g_{r,s}(y, A) g_{t,r}(x, dy).$$

The next lemma besides showing some continuity properties of the function $s \mapsto (G(t, s)f)(x)$ will be the key tool to prove the compactness of the operator $G(t, s)$ in Theorem 4.4. Let us consider the following strengthening of Hypothesis 2.1(iv).

Hypothesis 3.3. For every bounded interval $J \subset I$ there exist a positive function $\varphi = \varphi_J \in C^2(\mathbb{R}^d)$ and a real number $\lambda = \lambda_J$ such that

$$\lim_{|x| \to +\infty} \varphi(x) = +\infty \quad \text{and} \quad \big(\mathcal{A}_{-c}(t)\varphi\big)(x) - \lambda\varphi(x) \leqslant 0, \quad (t, x) \in J \times \mathbb{R}^d.$$

Lemma 3.4. *The following properties hold true*:

(i) *suppose that $f \in C^2_c(\mathbb{R}^d)$. Then,*

$$\big(G(t, s_1)f\big)(x) - \big(G(t, s_0)f\big)(x) = -\int_{s_0}^{s_1} \big(G(t, \sigma)\mathcal{A}(\sigma)f\big)(x)\, d\sigma, \tag{3.5}$$

for any $s_0 \leqslant s_1 \leqslant t$ and any $x \in \mathbb{R}^d$. In particular, the function $(G(t,\cdot)f)(x)$ is differentiable in I_t for any $x \in \mathbb{R}^d$ and

$$\frac{\partial}{\partial s}\big(G(t,s)f\big)(x) = -\big(G(t,s)\mathcal{A}(s)f\big)(x);$$

(ii) *let $f \in C_b^2(\mathbb{R}^d)$ be constant and positive outside a ball and assume Hypothesis 3.3. Then, for any $x \in \mathbb{R}^d$, the function $(G(t,\cdot)\mathcal{A}(\cdot)f)(x)$ is locally integrable in I_t and*

$$\big(G(t,s_1)f\big)(x) - \big(G(t,s_0)f\big)(x) \geqslant -\int_{s_0}^{s_1}\big(G(t,\sigma)\mathcal{A}(\sigma)f\big)(x)\,d\sigma, \tag{3.6}$$

for any $s_0 \leqslant s_1 \leqslant t$.

Proof. (i) Let us fix $f \in C_c^2(\mathbb{R}^d)$ and let n be sufficiently large such that $\mathrm{supp}(f) \subset B(0,n)$. By [1, Theorem 2.3(ix)]

$$\big(G_n^D(t,s_1)f\big)(x) - \big(G_n^D(t,s_0)f\big)(x) = -\int_{s_0}^{s_1}\big(G_n^D(t,r)\mathcal{A}(r)f\big)(x)\,dr, \tag{3.7}$$

for any $s_0 \leqslant s_1 \leqslant t$ and any $x \in \mathbb{R}^d$, where we recall that $(G_n^D(t,s))$ is the evolution operator associated with the Cauchy–Dirichlet problem (2.2). Since the function $(r,x) \mapsto (\mathcal{A}(r)f)(x)$ is bounded and continuous in $[s_0,s_1] \times \mathbb{R}^d$, taking Theorem 2.3(ii) into account, we can let $n \to +\infty$ in (3.7) and obtain (3.5).

(ii) Since any function which is constant and positive in a neighborhood of ∞ can be split into the sum of a compactly supported function and a positive constant, due to the above result we just need to consider the case when $f = \mathbb{1}$.

Being rather long, we split the proof into three steps. To lighten the notation, throughout the proof we denote by $\|\psi\|_{\infty,R}$ the sup-norm over the ball $B(0,R)$ of the continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$.

Step 1. We first assume that the potential c is bounded in $J \times \mathbb{R}^d$ for any bounded set $J \subset \bar{I}$. As usual, let $(G_n^N(t,s))$ be the evolution operator associated with the Cauchy–Neumann problem (2.3). As it is well known,

$$\big(G_n^N(t,s_1)f\big)(x) - \big(G_n^N(t,s_0)f\big)(x) = -\int_{s_0}^{s_1}\big(G_n^N(t,\tau)\mathcal{A}(\tau)f\big)(x)\,d\tau,$$

for any $f \in C_b^2(\mathbb{R}^d)$ such that $\frac{\partial f}{\partial \nu} = 0$ on $\partial B(0,n)$, any $s_0, s_1 \in I$ such that $s_0 \leqslant s_1 \leqslant t$ and any $x \in B(0,n)$. In particular, taking $f = \mathbb{1}$ yields

$$\big(G_n^N(t,s_1)\mathbb{1}\big)(x) - \big(G_n^N(t,s_0)\mathbb{1}\big)(x) = \int_{s_0}^{s_1}\big(G_n^N(t,\tau)c(\tau,\cdot)\big)(x)\,d\tau, \tag{3.8}$$

for any s_0, s_1, t and x as above. Theorem 2.3(i) shows that $(G_n^N(t,s_1)\mathbb{1})(x) - (G_n^N(t,s_0)\mathbb{1})(x)$ and $(G_n^N(t,\tau)c(\tau,\cdot))(x)$ tend to $(G(t,s_1)\mathbb{1})(x) - (G(t,s_0)\mathbb{1})(x)$ and $(G(t,\tau)c(\tau,\cdot))(x)$, respectively, as $n \to +\infty$. (Here the boundedness of c plays a crucial role.) Hence, taking the limit as $n \to +\infty$ in both the sides of (3.8) (and using again the boundedness of c) yields, by dominated convergence,

$$\big(G(t,s_1)\mathbb{1}\big)(x) - \big(G(t,s_0)\mathbb{1}\big)(x) = \int_{s_0}^{s_1}\big(G(t,\tau)c(\tau,\cdot)\big)(x)\,d\tau, \quad s_0 \leqslant s_1 < t,\ x \in \mathbb{R}^d. \tag{3.9}$$

Step 2. Let us now suppose that c is unbounded. Let us set $c_n(s,x) = (c(s,x) - c_0)\vartheta_n(x)$ for any $(s,x) \in I \times \mathbb{R}^d$, where c_0 is the infimum of the function c and $\vartheta_n \in C_c(\mathbb{R}^d)$ satisfies $\chi_{B(0,n)} \leqslant \vartheta_n \leqslant \chi_{B(0,n+1)}$ for any $n \in \mathbb{N}$. Clearly, each function c_n is nonnegative and belongs to $C(I; C_c(\mathbb{R}^d))$. Moreover, $c_n(s,x) \leqslant (c(s,x) - c_0)$ for any $(s,x) \in I \times \mathbb{R}^d$ and any $n \in \mathbb{N}$, and the sequence $(c_n(s,x))$ is increasing for any $(s,x) \in I \times \mathbb{R}^d$.

For any $n \in \mathbb{N}$, let $\mathcal{L}_n$ be the operator defined by

$$\big(\mathcal{L}_n(t)\psi\big)(x) = \sum_{i,j=1}^d q_{ij}(t,x)D_{ij}\psi(x) + \sum_{j=1}^d b_j(t,x)D_j\psi(x) - \big(c_n(t,x) + c_0\big)\psi(x),$$

for any $(t,x) \in I \times \mathbb{R}^d$. By Hypothesis 3.3, each operator $\mathcal{L}_n$ satisfies Hypothesis 2.1. Hence, we can associate an evolution operator $(G_n(t,s))$ with $\mathcal{L}_n$. Note that, for any nonnegative $f \in C_b(\mathbb{R}^d)$ and any $m,n \in \mathbb{N}$ such that $n < m$, the function $v = G_m(\cdot,s)f - G_n(\cdot,s)f$ satisfies the differential inequality $D_t v - \mathcal{L}_n v \leqslant 0$ and vanishes at $t = s$. The maximum principle in Proposition 2.2 then implies that $v \leqslant 0$ in $[s,+\infty) \times \mathbb{R}^d$, i.e.,

$$\big(G_m(t,s)f\big)(x) \leqslant \big(G_n(t,s)f\big)(x), \quad s \leqslant t,\ x \in \mathbb{R}^d.$$

In particular, for any fixed $t > s$ and $x \in \mathbb{R}^d$, the sequence $((G_n(t,s)f)(x))$ is nonincreasing. Hence, it converges to some function u as $n \to +\infty$. To show that $u = G(\cdot,s)f$, it suffices to use the same arguments as in the proof of Theorem 2.3. We leave the details to the reader.

Step 3. We now complete the proof. Writing (3.9) with G_n replacing G, we get

$$\begin{aligned}\big(G_n(t,s_1)\mathbb{1}\big)(x) - \big(G_n(t,s_0)\mathbb{1}\big)(x) &= \int_{s_0}^{s_1} \big(G_n(t,\tau)\big(c_n(\tau,\cdot)+c_0\big)\big)(x)\,d\tau \\ &\geqslant \int_{s_0}^{s_1} \big(G(t,\tau)c_n(\tau,\cdot)\big)(x)\,d\tau + c_0 \int_{s_0}^{s_1} \big(G_n(t,\tau)\mathbb{1}\big)(x)\,d\tau,\end{aligned}$$

for any $s_0, s_1 \in I$ such that $s_0 \leqslant s_1 \leqslant t$ and any $x \in \mathbb{R}^d$. Since the sequence (c_n) is increasing and $G(t,\tau)$ is a positive operator, we can apply Fatou lemma to pass to the limit as $n \to +\infty$ and get

$$\big(G(t,s_1)\mathbb{1}\big)(x) - \big(G(t,s_0)\mathbb{1}\big)(x) \geqslant \int_{s_0}^{s_1} \big(G(t,\tau)c(\tau,\cdot)\big)(x)\,d\tau.$$

This completes the proof. □

Corollary 3.5. *For any $f \in C_0(\mathbb{R}^d)$ the following properties are satisfied*:

(i) *the function $G(t,\cdot)f$ belongs to $C(I_t; C_b(\mathbb{R}^d))$ for any $t \in I$*;
(ii) *the function $(t,s,x) \mapsto (G(t,s)f)(x)$ is continuous in $\Lambda_I \times \mathbb{R}^d$.*

Proof. It suffices to prove the statements when $f \in C_c^\infty(\mathbb{R}^d)$. Indeed, the case when $f \in C_0(\mathbb{R}^d)$ follows by density, approximating f uniformly in $\mathbb{R}^d$ by a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ and taking into account that $G(\cdot,\cdot)f_n$ converges to $G(\cdot,\cdot)f$ uniformly in $K \times \mathbb{R}^d$ for any compact set $K \subset \Lambda_I$.

(i) Formula (3.5) shows that

$$\begin{aligned}\big\|G(t,s_1)f - G(t,s_0)f\big\|_\infty &\leqslant \sup_{r\in[s_0,s_1]} \big\|G(t,r)\mathcal{A}(r)f\big\|_\infty |s_1-s_0| \\ &\leqslant \sup_{r\in[s_0,s_1]} \big(e^{-c_0(t-r)}\big\|\mathcal{A}(r)f\big\|_\infty\big)|s_1-s_0|,\end{aligned}$$

for any $t \leqslant s_0 < s_1$, and this implies that the function $G(t,\cdot)f$ is locally Lipschitz continuous in $(-\infty,t]$ with values in $C_b(\mathbb{R}^d)$.

(ii) Using the classical Schauder estimates in [8, Theorem 3.5], we can show that, for any compact set $[a,b] \subset I$, any $m \in \mathbb{N}$ and any compact set $K \subset \mathbb{R}^d$, $\|G(\cdot,s)f\|_{C^{1+\alpha/2,2+\alpha}([s,s+m]\times K)}$ is bounded from above by a constant C_1 independent of $s \in [a,b]$. In particular, this shows that

$$\big|\big(G(t_2,s)f\big)(x) - \big(G(t_1,s)f\big)(x_0)\big| \leqslant C_1\big(|t_2-t_1| + |x-x_0|\big), \tag{3.10}$$

for any $t_1, t_2 \in [s,s+m]$, any $x, x_0 \in K$ and any $s \in [a,b]$.

Let $(t,s,x), (t_0,s_0,x_0) \in \Lambda_I \times \mathbb{R}^d$ with $s, s_0 \in [a,b]$ for some $[a,b] \subset I$. Assume that $s < s_0$; by (3.5) and (3.10) we can estimate

$$\begin{aligned}\big|\big(G(t,s)f\big)(x) - \big(G(t_0,s_0)f\big)(x_0)\big| &\leqslant \big|\big(G(t,s)f\big)(x) - \big(G(t_0,s)f\big)(x_0)\big| \\ &\quad + \big|\big(G(t_0,s)f\big)(x_0) - \big(G(t_0,s_0)f\big)(x_0)\big| \\ &\leqslant C_1\big(|t-t_0| + |x-x_0|\big) + C_2|s-s_0|,\end{aligned}$$

where $C_2 = \sup_{r\in[a,b]}(e^{-c_0(t_0-r)}\|\mathcal{A}(r)f\|_\infty)$. Hence,

$$\lim_{(t,s,x)\to(t_0,s_0^-,x_0)} \big(G(t,s)f\big)(x) = \big(G(t_0,s_0)f\big)(x_0).$$

Now, suppose that $s \geqslant s_0$ and $|t-t_0| \leqslant 1$. Then, $(t,s_0) \in \Lambda_I$ and

$$\begin{aligned}\big|\big(G(t,s)f\big)(x)-\big(G(t_0,s_0)f\big)(x_0)\big| &\leqslant \big|\big(G(t,s)f\big)(x)-\big(G(t,s_0)f\big)(x)\big| \\ &\quad + \big|\big(G(t,s_0)f\big)(x)-\big(G(t_0,s_0)f\big)(x_0)\big| \\ &\leqslant C_3|s-s_0| + \big|\big(G(t,s_0)f\big)(x)-\big(G(t_0,s_0)f\big)(x_0)\big|,\end{aligned}$$

where $C_3 := \max_{t\in[t_0-1,t_0+1]}\max_{s\in[a,b]}(e^{-c_0(t-r)}\|\mathcal{A}(r)f\|_\infty)$. Hence,

$$\lim_{(t,s,x)\to(t_0,s_0^+,x_0)} \big(G(t,s)f\big)(x) = \big(G(t_0,s_0)f\big)(x_0),$$

and the proof is completed. □

4. Compactness of the evolution operator in $C_b(\mathbb{R}^d)$

We now give sufficient conditions ensuring that the operator $G(t,s)$ is compact. As we already remarked in the introduction, in the case when $c \equiv 0$ (i.e., in the conservative case) a sufficient condition for $G(t,s)$ be compact in $C_b(\mathbb{R}^d)$ has been established in [14, Theorem 3.3]. For notational convenience, for any interval $J \subset I$, we set

$$\widetilde{\Lambda}_J := \big\{(t,s)\in J\times J:\ t>s\big\}. \tag{4.1}$$

To begin with let us give the definition of tightness for a one-parameter family of Borel measures. We stress that in the particular case of probability measures our definition agrees with the classical one.

Definition 4.1. Let $\mathcal{F}=\{\mu_\gamma:\ \gamma\in F\}$ be a family of finite Borel measures on $\mathbb{R}^d$. We say that $\mathcal{F}$ is tight if, for any $\varepsilon>0$, there exists $M>0$ such that $\mu_\gamma(\mathbb{R}^d\setminus B(0,M))\leqslant\varepsilon$ for any $\gamma\in F$.

We can now prove the following result.

Proposition 4.2. *Let $J\subset I$ be an interval. The following properties are equivalent*:

(i) *for every $(t,s)\in\widetilde{\Lambda}_J$, $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$;*
(ii) *for every $(t,s)\in\widetilde{\Lambda}_J$, the family of measures $\{g_{t,s}(x,dy):\ x\in\mathbb{R}^d\}$, defined in (3.4), is tight.*

Proof. (i) ⇒ (ii). Suppose that $G(t,s)$ is compact and consider a sequence (f_n) such that $\chi_{\mathbb{R}^d\setminus B(0,n+1)}\leqslant f_n\leqslant\chi_{\mathbb{R}^d\setminus B(0,n)}$ for any $n\in\mathbb{N}$. Clearly, f_n converges to 0 locally uniformly in $\mathbb{R}^d$, as $n\to+\infty$. Using the representation formula (3.1) it is easy to check that $G(t,s)f_n$ converges to 0 pointwise in $\mathbb{R}^d$. Since the operator $G(t,s)$ is compact and the sequence (f_n) is bounded, we can extract a subsequence (f_{n_k}) such that $G(t,s)f_{n_k}$ converges to 0 uniformly in $\mathbb{R}^d$. This is enough to infer that the whole sequence $(G(t,s)f_n)$ tends to 0, uniformly in $\mathbb{R}^d$, as $n\to+\infty$.

To complete the proof, it suffices to observe that

$$g_{t,s}\big(x,\mathbb{R}^d\setminus B(0,n)\big)=\big(G(t,s)\chi_{\mathbb{R}^d\setminus B(0,n)}\big)(x)\leqslant\big(G(t,s)f_{n-1}\big)(x),$$

for any $x\in\mathbb{R}^d$ and any $n\in\mathbb{N}$.

(ii) ⇒ (i). Fix $s,t\in I$ with $s<t$ and $r\in(s,t)$. Further, consider the family of operators S_n ($n\in\mathbb{N}$) defined as follows:

$$S_nf=G(t,r)\big(\chi_{B(0,n)}G(r,s)f\big),\quad f\in C_b\big(\mathbb{R}^d\big),\ n\in\mathbb{N}.$$

Since $G(t,r)$ is strong Feller (see Proposition 3.1(iii)), S_n is a bounded operator in $C_b(\mathbb{R}^d)$. Moreover,

$$\begin{aligned}\big|\big(G(t,s)f\big)(x)-(S_nf)(x)\big| &= \left|\int_{\mathbb{R}^d\setminus B(0,n)}\big(G(r,s)f\big)(y)g_{t,r}(x,dy)\right| \\ &\leqslant \big\|G(r,s)f\big\|_\infty g_{t,r}\big(x,\mathbb{R}^d\setminus B(0,n)\big),\end{aligned} \tag{4.2}$$

for any $x\in\mathbb{R}^d$, and the last side of (4.2) vanishes as $n\to+\infty$, uniformly with respect to $x\in\mathbb{R}^d$. Hence, to prove the assertion it suffices to show that each operator S_n is compact. This follows observing that the operator $f\mapsto(G(r,s)f)_{|\overline{B(0,n)}}$ is compact from $C_b(\mathbb{R}^d)$ into $C(\overline{B(0,n)})$ for any $n\in\mathbb{N}$ and, consequently, that the operator $f\mapsto\chi_{B(0,n)}G(r,s)f$ is compact from $C_b(\mathbb{R}^d)$ into $B_b(\mathbb{R}^d)$. Indeed, the interior Schauder estimates imply that, for any bounded family $\mathcal{F}\subset C_b(\mathbb{R}^d)$, the family $\mathcal{G}:=\{(G(r,s)f)_{|B(0,n)}:\ f\in\mathcal{F}\}$ is bounded in $C^{2+\alpha}(B(0,n))$. Therefore, $\mathcal{G}$ is equicontinuous and equibounded in $C(\overline{B(0,n)})$ by the Arzelà–Ascoli theorem, i.e., the operator $f\mapsto(G(r,s)f)_{|B(0,n)}$ is compact. Thus, S_n is compact as well. Being limit of compact operators, $G(t,s)$ is compact. □

In the following theorem we obtain a lower bound estimate for $g_{t,s}(x,\mathbb{R}^d)$ for every $t>s$ and every $x\in\mathbb{R}^d$, which is crucial for the proof of Theorem 4.4.

Proposition 4.3. *Assume that Hypothesis 3.3 holds. Let $J\subset I$ be an interval and suppose that there exist $\mu\in\mathbb{R}$, $R>0$ and a positive and bounded function $W\in C^2(\mathbb{R}^d\setminus B(0,R))$, such that $\inf_{x\in\mathbb{R}^d\setminus B(0,R)}W(x)>0$ and*

$$\big(\mathcal{A}(t)W\big)(x)-\mu W(x)\geqslant 0,\quad (t,x)\in J\times\mathbb{R}^d\setminus B(0,R). \tag{4.3}$$

Then, for any $s_0,T\in J$, such that $T>s_0$, there exists a positive constant C_{T,s_0} such that

$$\int_{\mathbb{R}^d} g_{t,s}(x,dy)\geqslant C_{T,s_0}, \tag{4.4}$$

for any $s,t\in\mathbb{R}$, with $s_0\leqslant s\leqslant t\leqslant T$, and any $x\in\mathbb{R}^d$.

Proof. We first assume that $c\geqslant 0$ and introduce the function v defined by

$$v(t,x)=e^{-\mu(t-s_0)}\big(G(t,s_0)\mathbb{1}\big)(x),\quad t\geqslant s_0,\ x\in\mathbb{R}^d.$$

Since $G(t,s_0)\mathbb{1}$ is everywhere positive in $\mathbb{R}^d$, the minimum of v over $[s_0,T]\times\overline{B(0,R)}$ is a positive constant, which we denote by κ.

Let $z:[s_0,T]\times\mathbb{R}^d\to\mathbb{R}$ be defined by $z(t,x)=v(t,x)-\gamma W(x)$ for any $(t,x)\in[s_0,T]\times\mathbb{R}^d$, where

$$\gamma=\kappa/\sup_{x\in\mathbb{R}^d\setminus B(0,R)}W(x).$$

Clearly, z belongs to $C_b([s_0,T]\times\mathbb{R}^d)\cap C^{1,2}((s_0,T]\times\mathbb{R}^d)$ and solves the following problem:

$$\begin{cases} D_tz(t,x)\geqslant\mathcal{A}_\mu(t)z(t,x), & t\in(s_0,T],\ x\in\mathbb{R}^d\setminus\overline{B(0,R)},\\ z(t,x)\geqslant 0, & t\in[s_0,T],\ x\in\partial B(0,R),\\ z(s_0,x)\geqslant 0, & \qquad\qquad x\in\mathbb{R}^d\setminus B(0,R).\end{cases}$$

The maximum principle in Proposition 2.2 implies that $z\geqslant 0$ in $[s_0,T]\times\mathbb{R}^d\setminus B(0,R)$ or, equivalently,

$$e^{-\mu(t-s_0)}\big(G(t,s_0)\mathbb{1}\big)(x)\geqslant\gamma W(x)\geqslant\gamma\inf_{y\in\mathbb{R}^d\setminus B(0,R)}W(y),$$

for any $t\in[s_0,T]$ and any $x\in\mathbb{R}^d\setminus B(0,R)$. It thus follows that $G(t,s_0)\mathbb{1}\geqslant C_{s_0,T}$ in $\mathbb{R}^d$, for any $s_0\leqslant t\leqslant T$, where

$$C_{s_0,T}=\min\big\{1,e^{\mu(T-s_0)}\big\}\min\Big\{\kappa,\gamma\inf_{y\in\mathbb{R}^d\setminus B(0,R)}W(y)\Big\}.$$

Let us now fix s such that $s_0<s<t$. From formula (3.9) we infer that the function $(G(t,\cdot)\mathbb{1})(x)$ is increasing. Therefore, $(G(t,s)\mathbb{1})(x)\geqslant(G(t,s_0)\mathbb{1})(x)\geqslant C_{s_0,T}$ for any $x\in\mathbb{R}^d$, and this accomplishes the proof in the case when $c\geqslant 0$, since by the representation formula (3.1) and (3.4), $g_{t,s}(x,\mathbb{R}^d)=(G(t,s)\mathbb{1})(x)$ for any $s,t\in I$, with $s<t$, and any $x\in\mathbb{R}^d$.

In the general case when $c_0<0$, let $(P(t,s))=(e^{c_0(t-s)}G(t,s))$ be the evolution operator associated with the second-order elliptic operator

$$\mathcal{A}_{-c_0}(t)=\sum_{i,j=1}^d q_{ij}(t,x)D_{ij}+\sum_{j=1}^d b_j(t,x)D_j-\big(c(t,x)-c_0\big).$$

Clearly, the operator $\mathcal{A}_{-c_0}$ satisfies Hypotheses 2.1(iv) and 3.3 with the same λ_J and φ_J. Moreover, it fulfills also assumption (4.3) with μ replaced with $c_0+\mu$. Hence, from the above arguments, it follows that for any s_0, T there exists a positive constant $C'_{s_0,T}$ such that $(P(t,s)\mathbb{1})(x)\geqslant C'_{s_0,T}$ for any $x\in\mathbb{R}^d$ and any $T\geqslant t\geqslant s\geqslant s_0$, and (4.4) follows with $C_{s_0,T}=C'_{s_0,T}$. The proof is complete. □

Adapting to our situation the technique in [16], we give a sufficient condition which ensures compactness of the operators $G(t,s)$ for $t>s$ in the non conservative case.

Theorem 4.4. *Assume that Hypothesis 3.3 is satisfied and there exist $R>0$, $d_1,d_2\in I$, with $d_1<d_2$, a positive function $\zeta\in C^2(\mathbb{R}^d)$, blowing up as $|x|\to+\infty$, and a convex increasing function $h:[0,+\infty)\to\mathbb{R}$ such that $1/h\in L^1(a,+\infty)$ for large a and*

$$\big(\mathcal{A}(s)\zeta\big)(x)\leqslant -h\big(\zeta(x)\big),\quad s\in[d_1,d_2],\ |x|\geqslant R. \tag{4.5}$$

Finally, let the assumptions of Proposition 4.3 hold true with $J=[d_1,d_2]$. Then, $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for any $(t,s)\in\Lambda_I$ such that $s\leqslant d_2$, $t\geqslant d_1$ and $t\neq s$.

Proof. Of course we can limit ourselves to proving the compactness of $G(t,s)$ for $(t,s) \in \widetilde{\Lambda}_{[d_1,d_2]}$ (see (4.1)) since for the other values of (t,s) it suffices to recall that $(G(t,s))$ is an evolution operator.

Let us first assume that $c \geqslant 0$. We will prove that for any $(t,s) \in \Lambda_{[d_1,d_2]}$ the family of measures $\{g_{t,s}(x,dy)\colon x \in \mathbb{R}^d\}$ is tight. First of all we prove that the function ζ is integrable with respect to every measure $g_{t,s}(x,dy)$ ($t > s$, $x \in \mathbb{R}^d$), so that $(G(t,s)\varphi)(x)$ is well defined for such t, s and x. For every $n \in \mathbb{N}$ choose $\psi_n \in C^2([0,+\infty))$ such that

(i) $\psi_n(r) = r$ for $r \in [0,n]$,
(ii) $\psi_n(r) = n + \frac{1}{2}$ for $r \geqslant n+1$,
(iii) $0 \leqslant \psi_n' \leqslant 1$ and $\psi_n'' \leqslant 0$.

Note that the previous conditions imply that $\psi_n'(r)r \leqslant \psi_n(r)$ for every $r \in [0,+\infty)$. The function $\zeta_n := \psi_n \circ \zeta$ belongs to $C^2(\mathbb{R}^d)$ and is constant outside a compact set for any $n \in \mathbb{N}$. By Lemma 3.4(ii), the differential inequality $\psi_n'(r)r \leqslant \psi_n(r)$ and the positivity of the function $G(t,s)\zeta$, we get

$$\begin{aligned}
\zeta_n(x) &\geqslant \zeta_n(x) - \big(G(t,s)\zeta_n\big)(x) \\
&\geqslant -\int_s^t \big(G(t,\sigma)\mathcal{A}(\sigma)\zeta_n\big)(x)\,d\sigma \\
&= -\int_s^t \big(G(t,\sigma)\big(\psi_n'(\zeta)\mathcal{A}(\sigma)\zeta + \psi_n''(\zeta)\langle Q\nabla\zeta, \nabla\zeta\rangle + c\big(\psi_n'(\zeta)\zeta - \zeta_n\big)\big)\big)(x)\,d\sigma \\
&\geqslant -\int_s^t \big(G(t,\sigma)\big(\psi_n'(\zeta)\mathcal{A}(\sigma)\zeta\big)\big)(x)\,d\sigma,
\end{aligned} \tag{4.6}$$

for any $x \in \mathbb{R}^d$. The right-hand side of (4.6) can be split into two parts as follows:

$$\begin{aligned}
\int_s^t \big(G(t,\sigma)\big(\psi_n'(\zeta)\mathcal{A}(\sigma)\zeta\big)\big)(x)\,d\sigma &= \int_s^t d\sigma \int_{A_+(\sigma)} \psi_n'\big(\zeta(y)\big)\big(\mathcal{A}(\sigma)\zeta\big)(y)g_{t,\sigma}(x,dy) \\
&\quad + \int_s^t d\sigma \int_{A_-(\sigma)} \psi_n'\big(\zeta(y)\big)\big(\mathcal{A}(\sigma)\zeta\big)(y)g_{t,\sigma}(x,dy),
\end{aligned} \tag{4.7}$$

where $A_+(\sigma) = \{y \in \mathbb{R}^d\colon (\mathcal{A}(\sigma)\zeta)(y) > 0\}$ and $A_-(\sigma) = \{y \in \mathbb{R}^d\colon (\mathcal{A}(\sigma)\zeta)(y) \leqslant 0\}$. Since $(\psi_n'(\zeta(y)))$ is nonnegative, increasing in n and converges to 1 for each $y \in \mathbb{R}^d$, the first integral in the right-hand side of (4.7) converges by the monotone convergence theorem to

$$\int_s^t d\sigma \int_{A_+(\sigma)} \big(\mathcal{A}(\sigma)\zeta\big)(y)g_{t,\sigma}(x,dy),$$

which is finite since the sets $A_+(\sigma)$ are equibounded in $\mathbb{R}^d$ (note that $(\mathcal{A}(\sigma)\zeta)(x)$ tends to $-\infty$ as $|x| \to +\infty$ uniformly with respect to $\sigma \in [d_1,d_2]$). Now, using (4.6) and (4.7) we get

$$\begin{aligned}
&-\int_s^t d\sigma \int_{A_-(\sigma)} \psi_n'\big(\zeta(y)\big)\big(\mathcal{A}(\sigma)\zeta\big)(y)g_{t,\sigma}(x,dy) \\
&\quad \leqslant \zeta_n(x) + \int_s^t d\sigma \int_{A_+(\sigma)} \psi_n'\big(\zeta(y)\big)\big(\mathcal{A}(\sigma)\zeta\big)(y)g_{t,\sigma}(x,dy).
\end{aligned}$$

Letting $n \to +\infty$ we deduce that the integral $\int_s^t d\sigma \int_{A_-(\sigma)} (\mathcal{A}(\sigma)\zeta)(y)g_{t,\sigma}(x,dy)$ is finite as well as the integral $\int_s^t (G(t,\sigma)\mathcal{A}(\sigma)\zeta)(x)\,d\sigma$. Moreover, since

$$\big(G(t,s)\zeta_n\big)(x) \leqslant \int_s^t \big(G(t,\sigma)\big(\psi_n'(\zeta)\mathcal{A}(\sigma)\zeta\big)\big)(x)\,d\sigma + \zeta_n(x),$$

letting $n \to +\infty$ we also deduce that $(G(t,s)\zeta)(x)$ is finite for every $(t,s) \in \Lambda_{[d_1,d_2]}$ and any $x \in \mathbb{R}^d$. Next, starting from the inequality

$$\big(G(t,s)\zeta_n\big)(x) - \big(G(t,r)\zeta_n\big)(x) \geqslant -\int_r^s \big(G(t,\sigma)\mathcal{A}(\sigma)\zeta_n\big)(x)\,d\sigma, \quad r<s<t,\ x \in \mathbb{R}^d$$

and arguing as above, we can show that

$$\big(G(t,s)\zeta\big)(x) - \big(G(t,r)\zeta\big)(x) \geqslant -\int_r^s \big(G(t,\sigma)\mathcal{A}(\sigma)\zeta\big)(x)\,d\sigma, \tag{4.8}$$

for every $r<s<t$ and $x \in \mathbb{R}^d$. Now, we prove that $(G(t,s)\zeta)(x)$ is bounded by a constant independent of x. Without loss of generality we can suppose that $(\mathcal{A}(s)\zeta)(x) \leqslant -h(\zeta(x))$, for any $s \in [d_1,d_2]$ and any $x \in \mathbb{R}^d$. Indeed, if this is not the case we replace h by $h-C$ for a suitable constant C. We can also assume that h vanishes at some point $x_h > 0$.

From the Jensen inequality for finite measures we get

$$h\bigg(\int_{\mathbb{R}^d} \zeta(y)g_{t,s}(x,dy)\bigg) \leqslant \frac{1}{g_{t,s}(x,\mathbb{R}^d)}\int_{\mathbb{R}^d} h\big(\zeta(y)\big)g_{t,s}(x,dy), \quad t>s,\ x\in\mathbb{R}^d,$$

since $0 < g_{t,s}(x,\mathbb{R}^d) = (G(t,s)\mathbb{1})(x) \leqslant 1$ for every $t \geqslant s$ and $x \in \mathbb{R}^d$, and h is increasing. We have thus obtained that

$$h\big(\big(G(t,s)\zeta\big)(x)\big) \leqslant \frac{1}{g_{t,s}(x,\mathbb{R}^d)}\big(G(t,s)(h\circ\zeta)\big)(x),$$

or, equivalently,

$$\big(G(t,s)(h\circ\zeta)\big)(x) \geqslant g_{t,s}\big(x,\mathbb{R}^d\big)h\big(\big(G(t,s)\zeta\big)(x)\big), \quad t>s,\ x\in\mathbb{R}^d.$$

Fix $s_0 < T$. Then, by Proposition 4.3 it follows that

$$\big(G(t,s)(h\circ\zeta)\big)(x) \geqslant C_{d_1,d_2}h\big(\big(G(t,s)\zeta\big)(x)\big) \quad \text{for each } d_1 \leqslant s \leqslant t \leqslant d_2,\ x\in\mathbb{R}^d.$$

Note that the function $(G(t,\cdot)(h\circ\zeta))(x)$ is integrable in $[d_1,t]$ for any $t \in (d_1,d_2]$ since it can be bounded from above by $-(G(t,\cdot)\mathcal{A}(\cdot)\zeta)(x)$.

Let us now fix $x \in \mathbb{R}^d$, $t \in [d_1,d_2]$ and define the function $\beta : [0,r_0) \to \mathbb{R}$, where $r_0 \in \mathbb{R}\cup\{+\infty\}$ satisfies $t - r_0 = \inf I$, by setting

$$\beta(r) := \big(G(t,t-r)\zeta\big)(x), \quad r \in [0,r_0).$$

Then, β is measurable since it is the limit of the sequence of the continuous functions $r \mapsto (G(t,t-r)\zeta_n)(x)$ (see Corollary 3.5).

Fix $b = t-d_1$. From estimate (4.8), the condition $(\mathcal{A}(s)\zeta)(x) \leqslant -h(\zeta(x))$, for any $s \in [d_1,d_2]$, and all the above remarks, we deduce that

$$\begin{aligned}
\beta(b)-\beta(0) &\leqslant -\int_{t-b}^{t} \big(G(t,\sigma)(h\circ\zeta)\big)(x)\,d\sigma \\
&\leqslant -C_{d_1,d_2}\int_{t-b}^{t} h\big(\big(G(t,\sigma)\zeta\big)(x)\big)\,d\sigma \\
&= -C_{d_1,d_2}\int_0^b h\big(\beta(\sigma)\big)\,d\sigma.
\end{aligned}$$

Let $y(\cdot) = y(\cdot;x)$ denote the solution of the following Cauchy problem

$$\begin{cases} y'(r) = -C_{d_1,d_2}h\big(y(r)\big), & r \geqslant 0, \\ y(0) = \zeta(x), \end{cases} \tag{4.9}$$

which is defined for all the positive times since h is increasing. Then, (i) $\beta(r) \leqslant y(r)$ for every $r \in [0,b]$ and (ii) $y(\cdot;x)$ is bounded from above in $[\delta,+\infty)$ for every $\delta > 0$, uniformly with respect to $x \in \mathbb{R}^d$, that is there exists $\bar{y} = \bar{y}(\delta) > 0$, independent of the initial datum $\zeta(x)$, such that $y(r;x) \leqslant \bar{y}$ for every $r \geqslant \delta$. To establish these properties it suffices to argue as in [14, Theorem 3.3] and [4, Theorem 5.1.5]. For the reader's convenience we provide here some details. To prove (i) one argues by contradiction and supposes that there exists $s_0 \in (0,b)$ such that $\beta(s_0) > y(s_0)$. Then, there exists an interval L containing s_0 where $\beta > y$. It suffices to observe that the inequality

$$\beta(s_2) - \beta(s_1) \leqslant -C_{d_1,d_2} \int_{s_1}^{s_2} h\big(\beta(\sigma)\big)\, d\sigma, \quad s_1, s_2 \in [0,d],$$

implies that the function $s \mapsto \beta(s) + C_{d_1,d_2} m s$, where $m := (\min_{\mathbb{R}_+} h)$, is decreasing. Thus,

$$\begin{aligned}\lim_{s \to s_0^-} \big(\beta(s) + C_{d_1,d_2} m s\big) &\geqslant \beta(s_0) + C_{d_1,d_2} m s_0 \\ &> y(s_0) + C_{d_1,d_2} m s_0 \\ &= \lim_{s \to s_0^-} \big(y(s) + C_{d_1,d_2} m s\big),\end{aligned}$$

so that $\beta > y$ in a left neighborhood of s_0. If we set $a = \inf L$, then $\beta(a) \leqslant y(a)$. We get to a contradiction observing that

$$\beta(s) - \beta(a) \leqslant -C_{d_1,d_2} \int_a^s h\big(\beta(\sigma)\big)\, d\sigma, \qquad y(s) - y(a) = -C_{d_1,d_2} \int_a^s h\big(y(\sigma)\big)\, d\sigma,$$

which yields

$$\beta(s) - y(s) \leqslant C_{d_1,d_2} \int_a^s \Big(h\big(y(\sigma)\big) - h\big(\beta(\sigma)\big)\Big)\, d\sigma, \quad s \in L,$$

which is a contradiction since the left-hand side is positive while the right-hand side is negative.

To prove (ii) we rewrite problem (4.9) into the following equivalent form:

$$-\int_{\zeta(x)}^{y(t;x)} \frac{dz}{h(z)} = C_{d_1,d_2} t. \tag{4.10}$$

Suppose that $\zeta(x) > x_h$ (where, we recall, x_h is the unique positive zero of h) and fix $\delta > 0$ and $t \geqslant \delta$. Since $1/h$ is integrable in a neighborhood of $+\infty$, using the above formula we conclude that

$$\int_{y(t;x)}^{+\infty} \frac{dz}{h(z)} \geqslant \int_{y(t;x)}^{\zeta(x)} \frac{dz}{h(z)} \geqslant C_{d_1,d_2} t \geqslant C_{d_1,d_2} \delta. \tag{4.11}$$

Since h is convex, $1/h$ is not integrable in a right neighborhood of x_h. Therefore, there exists a unique $M > x_h$ such that

$$\int_M^{+\infty} \frac{dz}{h(z)} = C_{d_1,d_2} \delta. \tag{4.12}$$

From (4.11) and (4.12) it follows that $y(t;x) \leqslant M$ for any $t \geqslant \delta$.

Suppose now that $\zeta(x) < x_h$. Then, from (4.10) it follows that $y(t;x) \leqslant x_h$ for any $t \geqslant \delta$. The proof of property (ii) is now complete.

The properties (i) and (ii) now imply that $(G(t,t-r)\zeta)(x) \leqslant \bar{y}$ for every $r \in [\delta, t-d_1]$. Let $R > 0$ and assume that $s \in [d_1, t-\delta]$. Then, $(G(t,s)\zeta)(x) \leqslant \bar{y}$. Hence,

$$\begin{aligned} g_{t,s}\big(x, \mathbb{R}^d \setminus B(0,R)\big) &= \int_{\mathbb{R}^d \setminus B(0,R)} g_{t,s}(x,dy) \\ &\leqslant \frac{1}{\inf\{\zeta(y)\colon |y| \geqslant R\}} \int_{\mathbb{R}^d \setminus B(0,R)} \zeta(y) g_{t,s}(x,dy) \\ &\leqslant \frac{(G(t,s)\zeta)(x)}{\inf\{\zeta(y)\colon |y| \geqslant R\}} \\ &\leqslant \frac{\bar{y}}{\inf\{\zeta(y)\colon |y| \geqslant R\}}. \end{aligned}$$

Since $\inf\{\zeta(y)\colon |y| \geqslant R\}$ tends to $+\infty$ as $R \to +\infty$, it follows that, for any $\varepsilon > 0$, $g_{t,s}(x, \mathbb{R}^d \setminus B(0,R)) \leqslant \varepsilon$, for any $x \in \mathbb{R}^d$, if R is sufficiently large and $s \in [d_1, t-\delta]$. The arbitrariness of $\delta > 0$ allows us to conclude through Proposition 4.2.

Let us now consider the general case when the infimum c_0 of c is negative. We introduce the evolution operator $(P(t,s)) = (e^{c_0(t-s)}G(t,s))$ which is associated with the elliptic operator $\mathcal{A}_{-c_0}$. Note that $\mathcal{A}_{-c_0}$ satisfies assumption (4.5) with h replaced by $h - c_0$. Moreover, $\mathcal{A}_{-c_0}(t)W - (c_0+\mu)W \geqslant 0$ in $[d_1, d_2] \times \mathbb{R}^d \setminus B(0,R)$. Since Hypothesis 3.3 is trivially fulfilled, we conclude that the operator $P(t,s)$ is compact for any $s, t \in [d_1, d_2]$ with $s < t$. As a byproduct $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$ for the same values of s and t. This accomplishes the proof. $\square$

Remark 4.5. In the conservative case treated in [14], the existence of the function W as in Proposition 4.3 is not needed, since $g_{t,s}(x, \mathbb{R}^d) = 1$ for every $t > s$ and every $x \in \mathbb{R}^d$. Hence, (4.4) is trivially satisfied.

Remark 4.6. Theorem 4.4 gives a sufficient condition for the compactness of the operator $G(t,s)$. Assuming only Hypothesis 2.1, in general one does not expect that the operator $G(t,s)$ is compact. A very easy counterexample is provided by the operator $\mathcal{A}u = \Delta u - cu$, where $c \in C^{\alpha}_b(\mathbb{R}^{1+d})$ for some $\alpha \in (0,1)$. A straightforward comparison argument shows that the associated evolution operator $(G(t,s))$ is positive and satisfies

$$e^{-c_1(t-s)}T(t-s)f \leqslant G(t,s)f \leqslant e^{-c_0(t-s)}T(t-s)f, \quad f \in C_b\big(\mathbb{R}^d\big),\ f \geqslant 0, \tag{4.13}$$

for any $s < t$, where c_0 and c_1 are, respectively, the infimum and the supremum of the function c, and $(T(t))$ is the Gauss–Weierstrass semigroup. It is well known that this semigroup preserves $C_0(\mathbb{R}^d)$. Take a sequence of (smooth) compactly supported nonnegative functions (f_n) converging to $\mathbb{1}$ locally uniformly in $\mathbb{R}^d$. If $G(t,s)$ were compact, up to a subsequence, $G(t,s)f_n$ would converge to $G(t,s)\mathbb{1}$ uniformly in $\mathbb{R}^d$ (take Proposition 3.1(ii) into account), which, in view of (4.13), satisfies $G(t,s)\mathbb{1} \geqslant e^{-c_1(t-s)}$ and, hence, it does not belong to $C_0(\mathbb{R}^d)$. This is a contradiction since $G(t,s)f_n \in C_0(\mathbb{R}^d)$ for any $n \in \mathbb{N}$.

4.1. A consequence of Proposition 4.3 and the compactness of $G(t,s)$

Let us prove the following result.

Theorem 4.7. *Let the assumptions of Proposition* 4.3 *be satisfied. Further, assume that, for some $s, t \in J$ with $s < t$, the operator $G(t,s)$ is compact in $C_b(\mathbb{R}^d)$. Then, $G(t,s)$ preserves neither $C_0(\mathbb{R}^d)$ nor $L^p(\mathbb{R}^d)$ ($p \in [1, +\infty)$).*

Proof. Let (f_n) be a sequence of smooth functions such that $\chi_{B(0,n)} \leqslant f_n \leqslant \chi_{B(0,2n)}$ for any $n \in \mathbb{N}$, and fix $s, t \in I$ with $s < t$. From formula (3.1) and the dominated convergence theorem, it follows immediately that $G(t,s)f_n$ converges pointwise in $\mathbb{R}^d$ to $G(t,s)\mathbb{1}$ as $n \to +\infty$. Since $G(t,s)$ is a compact operator, $G(t,s)f_n$ actually converges uniformly in $\mathbb{R}^d$ to $G(t,s)\mathbb{1}$. Since $G(t,s)$ is bounded in $C_b(\mathbb{R}^d)$, if it preserved $C_0(\mathbb{R}^d)$ the function $G(t,s)\mathbb{1}$ would tend to 0 as $|x| \to +\infty$, but this is not the case. Indeed, formula (4.4) shows that $G(t,s)\mathbb{1}$ is bounded from below by a positive constant.

To prove that $G(t,s)$ does not preserve $L^p(\mathbb{R}^d)$, we denote by K any positive constant such that $g_{t,s}(x, \mathbb{R}^d) \geqslant K$ for any $x \in \mathbb{R}^d$. By Proposition 4.2, we can fix $R > 0$ such that $g_{t,s}(x, \mathbb{R}^d \setminus B(0,R)) \leqslant K/2$. By difference it follows that

$$\big(G(t,s)\chi_{B(0,R)}\big)(x) = g_{t,s}\big(x, \mathbb{R}^d\big) - g_{t,s}\big(x, \mathbb{R}^d \setminus B(0,R)\big) \geqslant \frac{K}{2}, \quad x \in \mathbb{R}^d.$$

Hence, $G(t,s)\chi_{B(0,R)}$ does not belong to $L^p(\mathbb{R}^d)$. $\square$

4.2. An extension of Corollary 3.5 to $C_b(\mathbb{R}^d)$

An insight in the proof of Theorem 4.4 shows that, if $c \geqslant 0$ and

$$\big(\mathcal{A}(t)\zeta\big)(x) \leqslant -h\big(\zeta(x)\big), \quad t \in J,\ |x| \geqslant R, \tag{4.14}$$

for some interval $J \subset I$ and some $R > 0$, then

$$M_{J,\rho,\delta} = \sup_{\substack{(t,s)\in\Lambda_J,\, t-s>\delta\\ |x|\leqslant\rho}} \big(G(t,s)\zeta\big)(x) < +\infty, \tag{4.15}$$

for any $\delta, \rho > 0$.

Actually, as in [12], slightly modifying the proof, we can improve (4.15), removing the condition $t - s \geqslant \delta$. For this purpose, in fact, we just need a weaker assumption than (4.14). More precisely we will assume that the following hypothesis is satisfied.

Hypothesis 4.8. For every bounded interval $J \subset I$ there exist a positive function $\varphi = \varphi_J \in C^2(\mathbb{R}^d)$ diverging to $+\infty$ as $|x| \to +\infty$ and a positive constant M_J such that

$$\big(\mathcal{A}(t)\varphi\big)(x) \leqslant M_J, \quad (t,x) \in J \times \mathbb{R}^d.$$

Proposition 4.9. *Let $c \geqslant 0$ and assume that Hypothesis* 4.8 *holds. Then, $G(\cdot,\cdot)\varphi$ is bounded in $\Lambda_J \times B(0,\rho)$ for every $\rho > 0$.*

Proof. We can repeat the proof of Theorem 4.4 until formula (4.6), so that we have

$$\varphi_n(x) - \big(G(t,s)\varphi_n\big)(x) \geqslant -\int_s^t d\sigma \int_{\mathbb{R}^d} \psi_n'\big(\varphi(y)\big)\big(\mathcal{A}(\sigma)\varphi\big)(y) g_{t,\sigma}(x,dy),$$

for any $(t,s) \in \Lambda_J$ and any $x \in \mathbb{R}^d$, where $\varphi_n = \psi_n \circ \varphi$. Since $(\psi_n'(\varphi(y)))$ is nonnegative using the assumptions we get

$$\varphi_n(x) - \big(G(t,s)\varphi_n\big)(x) \geqslant -M_J \int_s^t d\sigma \int_{\mathbb{R}^d} \psi_n'\big(\varphi(y)\big) g_{t,\sigma}(x,dy). \tag{4.16}$$

Letting $n \to +\infty$ in (4.16) we get

$$\big(G(t,s)\varphi\big)(x) \leqslant \varphi(x) + M_J(t-s),$$

for any $s,t \in J$, such that $s \leqslant t$, and any $x \in \mathbb{R}^d$. The claim follows. □

As a consequence of Proposition 4.9 we obtain the following result.

Proposition 4.10. *Assume that Hypotheses* 3.3 *and* 4.8 *hold. Then, for every bounded interval $J \subset I$ and every $R > 0$, the family of measures $\{g_{t,s}(x,dy)\colon (t,s,x) \in \Lambda_J \times \overline{B(0,R)}\}$ is tight.*

Proof. In the case when $c_0 \geqslant 0$ the proof is similar to that of [12, Lemma 3.5]. If $c_0 < 0$ we can consider the evolution operator $(P(t,s)) = (e^{c_0(t-s)}G(t,s))$ associated with the elliptic operator $\mathcal{A}_{-c_0}$, whose potential term is nonpositive and satisfies Hypotheses 3.3 and 4.8. Then, the family of measures $\{p_{t,s}(x,dy)\colon (t,s,x) \in \Lambda_J \times \overline{B(0,R)}\}$ associated with $(P(t,s))$ satisfies the claim as well as the family $\{g_{t,s}(x,dy)\colon (t,s,x) \in \Lambda_J \times \overline{B(0,R)}\}$ since $p_{t,s}(x,dy) = e^{c_0(t-s)}g_{t,s}(x,dy)$ for every $(t,s,x) \in \Lambda_I \times \mathbb{R}^d$. □

The following result allows us to extend the continuity property of the function $G(\cdot,\cdot)f$, stated in Corollary 3.5 for $f \in C_0(\mathbb{R}^d)$, to the case when f is merely bounded and continuous in $\mathbb{R}^d$.

Proposition 4.11. *Assume that Hypotheses* 3.3 *and* 4.8 *hold. Let $(f_n) \subset C_b(\mathbb{R}^d)$ be a bounded sequence converging to $f \in C_b(\mathbb{R}^d)$ locally uniformly in $\mathbb{R}^d$. Then, $G(\cdot,\cdot)f_n$ converges to $G(\cdot,\cdot)f$ locally uniformly in $\Lambda_I \times \mathbb{R}^d$.*

Proof. The proof can be obtained as the proof of [12, Proposition 3.6], taking Proposition 4.10 into account. □

Theorem 4.12. *Under the assumptions of Proposition* 4.11, *the function $G(\cdot,\cdot)f$ is continuous in $\Lambda_I \times \mathbb{R}^d$, for every $f \in C_b(\mathbb{R}^d)$.*

Proof. Let $f \in C_b(\mathbb{R}^d)$, by Proposition 4.11 we can find a sequence of bounded functions $f_n \in C_c^\infty(\mathbb{R}^d)$ converging to f locally uniformly in $\mathbb{R}^d$ such that $G(\cdot,\cdot)f_n$ converges to $G(\cdot,\cdot)f$ locally uniformly. Since any function $G(\cdot,\cdot)f_n$ is continuous in $\Lambda_I \times \mathbb{R}^d$ for any $n \in \mathbb{N}$ by Corollary 3.5, the assertion follows at once. □

5. Invariance of $C_0(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$

In Section 4.1 we have obtained some conditions which imply that neither $C_0(\mathbb{R}^d)$ nor $L^p(\mathbb{R}^d)$ is preserved by $G(t,s)$. Here, we provide sufficient conditions for $C_0(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$ be preserved by $G(t,s)$.

5.1. Invariance of $C_0(\mathbb{R}^d)$

Theorem 5.1. *Fix $a,b \in I$ such that $a < b$. Assume that there exist a strictly positive function $V \in C^2(\mathbb{R}^d)$ and $\lambda_0 > 0$ such that $\lim_{|x|\to+\infty} V(x) = 0$ and $\lambda_0 V(x) - (\mathcal{A}(t)V)(x) \geqslant 0$ for every $(t,x) \in [a,b] \times \mathbb{R}^d$. Then, $G(t,s)$ preserves $C_0(\mathbb{R}^d)$ for any $(t,s) \in \Lambda_{[a,b]}$.*

Proof. Fix $s \in [a,b]$. It suffices to prove the statement for $f \in C_c(\mathbb{R}^d)$ since we may approximate an arbitrary $f \in C_0(\mathbb{R}^d)$ by a sequence $(f_n) \subset C_c(\mathbb{R}^d)$ with respect to the sup-norm in $\mathbb{R}^d$, and $G(t,s)f_n$ converges uniformly to $G(t,s)f$ for every $t \geqslant s$. It is not restrictive to suppose $f \geqslant 0$ otherwise we consider its positive and negative part. Fix $R > 0$, assume that supp $f \subset$

$B(0,R)$ and consider the unique bounded classical solution u of the Cauchy problem (2.1). Let $\delta = \inf_{x\in B(0,R)} V(x) > 0$ and $z(t,x) = e^{-\lambda_0(t-s)}u(t,x) - \delta^{-1}\|f\|_\infty V(x)$. Then, the function $z \in C_b([s,b]\times\mathbb{R}^d)\cap C^{1,2}((s,b]\times\mathbb{R}^d)$ satisfies

$$\begin{cases} D_t z(t,x) - \mathcal{A}_{\lambda_0}(t)z(t,x) \leqslant 0, & (t,x)\in(s,b]\times\mathbb{R}^d,\\ z(s,x)\leqslant 0, & x\in\mathbb{R}^d. \end{cases}$$

Therefore, applying Proposition 2.2 (with $\mathcal{A}$ replaced with $\mathcal{A}_{\lambda_0}$) we get $z \leqslant 0$, i.e.,

$$0 \leqslant u(t,x) \leqslant e^{\lambda_0(t-s)}\delta^{-1}\|f\|_\infty V(x), \quad s\leqslant t\leqslant b,\ x\in\mathbb{R}^d, \tag{5.1}$$

which implies that $u \in C_0(\mathbb{R}^d)$. $\square$

In the autonomous case it is known that, under Hypothesis 2.1, $T(t)f$ tends to f uniformly in $\mathbb{R}^d$ as $t\to 0^+$, for any $f\in C_0(\mathbb{R}^d)$, where $(T(t))$ is the semigroup in $C_b(\mathbb{R}^d)$ associated with the operator $\mathcal{A}$. Typically this result is proved, first for functions $f\in C_c^2(\mathbb{R}^d)$ and, then, is extended by density to any $f\in C_0(\mathbb{R}^d)$. In the former case when $f\in C_c^2(\mathbb{R}^d)$, one can show that

$$\big(T(t)f\big)(x) - f(x) = \int_0^t \big(T(s)\mathcal{A}f\big)(x)\,ds, \quad t>0,\ x\in\mathbb{R}^d, \tag{5.2}$$

since $\mathcal{A}$ commutes with the semigroup $(T(t))$. Using this formula it is immediate to show that $T(t)f$ tends to f, uniformly in $\mathbb{R}^d$, as $t\to 0^+$. (See e.g. [4, Proposition 2.2.7], [15, Proposition 4.3] for further details.) When $\mathcal{A}$ depends on t, in general $G(t,s)$ does not commute with $\mathcal{A}$. This prevents us to write a formula similar to (5.2). Hence, it is not clear, in general, if the function $G(t,s)f$ tends to f uniformly in $\mathbb{R}^d$ as $t\to s^+$, when $f\in C_0(\mathbb{R}^d)$. This turns out to be the case when the assumptions of Theorem 5.1 are satisfied, as the following corollary shows.

Corollary 5.2. *Under the assumptions of Theorem* 5.1*, for any $f\in C_0(\mathbb{R}^d)$, the function $(t,s)\mapsto G(t,s)f$ is continuous in $\Lambda_{[a,b]}$ with values in $C_0(\mathbb{R}^d)$. In particular, if for any $I\ni a<b$ there exist $\lambda=\lambda_{a,b}$ and $V=V_{a,b}$, with the properties in Theorem* 5.1*, then the restriction of $(G(t,s))$ to $C_0(\mathbb{R}^d)$ gives rise to a strongly continuous evolution operator.*

Proof. We split the proof in two steps.

Step 1. Here, we prove that the map $G(\cdot,s_0)f$ is continuous in $[s_0,b]$ for any $s_0\in[a,b)$ and any $f\in C_0(\mathbb{R}^d)$. We first assume that $f\in C_c(\mathbb{R}^d)$ and prove the continuity of the map $G(\cdot,s_0)f$ at t_0. It is not restrictive to assume that f is nonnegative and it does not identically vanish in $\mathbb{R}^d$. Indeed, if this is not the case we split $G(t,s)f = G(t,s)f^+ - G(t,s)f^-$, where, as usual, f^+ and f^- are the positive and negative parts of f, respectively.

Let R and δ be as in the proof of Theorem 5.1, fix $\varepsilon>0$ and choose $\tilde{R}>0$ such that $V(x)\leqslant\varepsilon\delta e^{-\lambda_0(b-a)}\|f\|_\infty^{-1}$ for any $|x|\geqslant\tilde{R}$. Then, estimate (5.1) implies that $(G(t,s_0)f)(x)\leqslant\varepsilon$ for any $|x|\geqslant\tilde{R}$ and any $t\in[s_0,b]$. Therefore,

$$\begin{aligned} \big\|G(t,s_0)f - G(t_0,s_0)f\big\|_\infty &\leqslant \sup_{|x|\leqslant\tilde{R}}\big|\big(G(t,s_0)f\big)(x) - \big(G(t_0,s_0)f\big)(x)\big| \\ &\quad + \sup_{|x|\geqslant\tilde{R}}\big|\big(G(t,s_0)f\big)(x) - \big(G(t_0,s_0)f\big)(x)\big| \\ &\leqslant \sup_{|x|\leqslant\tilde{R}}\big|\big(G(t,s_0)f\big)(x) - \big(G(t_0,s_0)f\big)(x)\big| + 2\varepsilon. \end{aligned} \tag{5.3}$$

Since the function $(t,x)\mapsto(G(t,s_0)f)(x)$ is continuous in $[s_0,+\infty)\times\mathbb{R}^d$, $G(t,s_0)f$ tends to $G(t_0,s_0)f$ as $t\to t_0$, locally uniformly in $\mathbb{R}^d$. Hence, from (5.3) we obtain that

$$\limsup_{t\to t_0}\big\|G(t,s_0)f - G(t_0,s_0)f\big\|_\infty \leqslant 2\varepsilon.$$

The arbitrariness of $\varepsilon>0$ allows us to conclude that $G(t,s_0)f$ tends to $G(t_0,s_0)f$ as $t\to t_0$, uniformly in $\mathbb{R}^d$.

The continuity of $G(\cdot,s_0)f$ when $f\in C_0(\mathbb{R}^d)$ follows approximating f by a sequence of compactly supported functions and taking (2.4) into account.

Step 2. Here we complete the proof. We assume that $f\in C_c^2(\mathbb{R}^d)$. As in Step 1, the general case then will follow by a standard approximation argument and estimate (2.4).

So, let us fix $f\in C_c^2(\mathbb{R}^d)$, $(s_0,t_0),(s,t)\in\Lambda_{[a,b]}$ and prove the continuity of the map $G(\cdot,\cdot)f$ at (t_0,s_0). For this purpose we observe that, if $t>t_0$, we can split

$$G(t,s)f - G(t_0,s_0)f = G(t,t_0)\big(G(t_0,s)f - G(t_0,s_0)f\big) + G(t,t_0)G(t_0,s_0)f - G(t_0,s_0)f.$$

Therefore, using (2.4) we can estimate

$$\begin{aligned}\big\|G(t,s)f-G(t_0,s_0)f\big\|_\infty &\leqslant e^{-c_0(t-t_0)}\big\|G(t_0,s)f-G(t_0,s_0)f\big\|_\infty\\ &\quad+\big\|G(t,t_0)G(t_0,s_0)f-G(t_0,s_0)f\big\|_\infty.\end{aligned}$$

Step 1 and Corollary 3.5(i) show that the right-hand side of the previous inequality tends to 0 as $s\to s_0$ and $t\to t_0^+$.

We now assume that $t<t_0$ and split

$$\begin{aligned}G(t,s)f-G(t_0,s_0)f&=G(t,s)f-G(t,s_0)f-\big(G(t_0,t)G(t_0,s_0)f-G(t_0,s_0)f\big)\\ &\quad-G(t_0,t)\big(G(t,s_0)f-G(t_0,s_0)f\big)+\big(G(t,s_0)f-G(t_0,s_0)f\big).\end{aligned}$$

Hence,

$$\begin{aligned}\big\|G(t,s)f-G(t_0,s_0)f\big\|&\leqslant\big\|G(t,s)f-G(t,s_0)f\big\|_\infty+\big\|G(t_0,t)G(t_0,s_0)f-G(t_0,s_0)f\big\|_\infty\\ &\quad+e^{c_0(t-t_0)}\big\|G(t,s_0)f-G(t_0,s_0)f\big\|_\infty+\big\|G(t,s_0)f-G(t_0,s_0)f\big\|_\infty.\end{aligned}$$

Using Step 1 and Corollary 3.5(i), we conclude that the right-hand side of the previous inequality tends to 0 as $t\to t_0^-$ and $s\to s_0$. This completes the proof. □

5.2. Invariance of $L^p(\mathbb{R}^d)$

We now study the invariance of $L^p(\mathbb{R}^d)$ under the action of the operator $G(t,s)$.

Theorem 5.3. *Fix $a,b\in I$, with $a<b$. Suppose that the diffusion coefficients q_{ij} and the drift coefficients b_j $(i,j=1,\dots,d)$ are continuously differentiable in $[a,b]\times\mathbb{R}^d$, with respect to the spatial variables, and the derivatives $D_{ij}q_{ij}$ $(i,j=1,\dots,d)$ exist in $[a,b]\times\mathbb{R}^d$. Further, assume that there exists $K>0$ such that*

$$c(t,x)+\mathrm{div}_x\,\beta(t,x)\geqslant-K,\quad (t,x)\in[a,b]\times\mathbb{R}^d,\tag{5.4}$$

where

$$\beta_i(t,x)=b_i(t,x)-\sum_{j=1}^d D_jq_{ij}(t,x),\quad (t,x)\in I\times\mathbb{R}^d,\ i=1,\dots,d.\tag{5.5}$$

Then, for every $1\leqslant p<+\infty$, $L^p(\mathbb{R}^d)$ is invariant under $G(t,s)$ for any $(t,s)\in\Lambda_{[a,b]}$. Moreover,

$$\big\|G(t,s)f\big\|_{L^p(\mathbb{R}^d)}\leqslant e^{K_p(t-s)}\|f\|_{L^p(\mathbb{R}^d)},\quad a\leqslant s\leqslant t\leqslant b,\tag{5.6}$$

where $pK_p=K-(p-1)c_0$.

Proof. Fix $s\in[a,b]$. We prove the assertion for nonnegative $f\in C_c^\infty(\mathbb{R}^d)$. The density of $C_c^\infty(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$ $(p\in[1,+\infty))$ combined with the estimate $|G(t,s)f|\leqslant G(t,s)|f|$ (see (3.1)) then allows us to extend the result to any $f\in L^p(\mathbb{R}^d)$.

Let $u=G(\cdot,s)f$ and, for any $n\in\mathbb{N}$, let $u_n=G_n^D(\cdot,s)f$ be the classical solution of the Cauchy–Dirichlet problem (2.2). By Theorem 2.3(ii), u_n is nonnegative in $[s,+\infty)\times\mathbb{R}^d$ and therein converges to u pointwise as $n\to+\infty$.

Let us prove that

$$\big\|u_n(t,\cdot)\big\|_{L^p(B(0,n))}\leqslant e^{K_p(t-s)}\big\|u_n(s,\cdot)\big\|_{L^p(B(0,n))}=e^{K_p(t-s)}\|f\|_{L^p(B(0,n))},\tag{5.7}$$

for any $t\in[s,b]$ and any $n\in\mathbb{N}$ such that $\mathrm{supp}(f)\subset B(0,n)$. We first assume that $p\neq1$ and set $u_n^\varepsilon=u_n+\varepsilon$. Then

$$\frac{d}{dt}\big\|u_n^\varepsilon(t,\cdot)\big\|^p_{L^p(B(0,n))}=p\int\limits_{B(0,n)}\big(u_n^\varepsilon(t,\cdot)\big)^{p-1}\mathcal{A}(t)u_n(t,\cdot)\,dx.$$

Integrating by parts and observing that $u_n^\varepsilon\equiv\varepsilon$ on $\partial B(0,n)$, we get

$$\begin{aligned}&\int\limits_{B(0,n)}\big(u_n^\varepsilon(t,\cdot)\big)^{p-1}\mathcal{A}(t)u_n(t,\cdot)\,dx\\ &\quad=\varepsilon^{p-1}\int\limits_{\partial B(0,n)}\big\langle Q(t,\cdot)\nabla_xu_n(t,\cdot),\nu\big\rangle d\mathcal{H}^{n-1}(x)\\ &\qquad-(p-1)\int\limits_{B(0,n)}\big(u_n^\varepsilon(t,\cdot)\big)^{p-2}\big\langle Q(t,\cdot)\nabla_xu_n(t,\cdot),\nabla_xu_n(t,\cdot)\big\rangle dx\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p}\int_{B(0,n)}\big\langle\beta(t,\cdot),\nabla_x\big(u_n^\varepsilon(t,\cdot)\big)^p\big\rangle\,dx-\int_{B(0,n)}c(t,\cdot)\big(u_n^\varepsilon(t,\cdot)\big)^p\,dx\\
&+\varepsilon\int_{B(0,n)}c(t,\cdot)\big(u_n^\varepsilon(t,\cdot)\big)^{p-1}\,dx\\
\leqslant{}&\varepsilon^{p-1}\int_{\partial B(0,n)}\big\langle Q(t,\cdot)\nabla_x u_n(t,\cdot),\nu\big\rangle\,d\mathcal{H}^{n-1}(x)+\frac{\varepsilon^p}{p}\int_{\partial B(0,n)}\big\langle\beta(t,\cdot),\nu\big\rangle\,d\mathcal{H}^{n-1}(x)\\
&-\frac{1}{p}\int_{B(0,n)}\big(u_n^\varepsilon(t,\cdot)\big)^p\big(pc(t,\cdot)+\operatorname{div}_x\beta(t,\cdot)\big)\,dx\\
&+\varepsilon\int_{B(0,n)}c(t,\cdot)\big(u_n^\varepsilon(t,\cdot)\big)^{p-1}\,dx,
\end{aligned}
\tag{5.8}
$$

where $\nu=\nu(x)$ is the outward unit normal at $x\in\partial B(0,n)$. We now observe that

$$
\begin{aligned}
pc(t,\cdot)+\operatorname{div}_x\beta(t,\cdot)&=p\big(c(t,\cdot)-c_0\big)+pc_0+\operatorname{div}_x\beta(t,\cdot)\\
&\geqslant\big(c(t,\cdot)-c_0\big)+\operatorname{div}_x\beta(t,\cdot)+pc_0\geqslant -pK_p.
\end{aligned}
$$

Hence, from (5.8) we get

$$
\begin{aligned}
&\int_{B(0,n)}\big(u_n^\varepsilon(t,\cdot)\big)^{p-1}\mathcal{A}(t)u_n(t,\cdot)\,dx\\
&\quad\leqslant\varepsilon^{p-1}\int_{\partial B(0,n)}\big\langle Q(t,\cdot)\nabla_x u_n(t,\cdot),\nu\big\rangle\,d\mathcal{H}^{n-1}(x)+\frac{\varepsilon^p}{p}\int_{\partial B(0,n)}\big\langle\beta(t,\cdot),\nu\big\rangle\,d\mathcal{H}^{n-1}(x)\\
&\qquad+K_p\int_{B(0,n)}\big(u_n^\varepsilon(t,\cdot)\big)^p\,dx+\varepsilon\int_{B(0,n)}c(t,\cdot)\big(u_n^\varepsilon(t,\cdot)\big)^{p-1}\,dx,
\end{aligned}
\tag{5.9}
$$

for any $t\in[s,b]$. If we set

$$
\begin{aligned}
g_n^{\varepsilon,p}(t):={}&p\varepsilon^{p-1}\int_{\partial B(0,n)}\big\langle Q(t,\cdot)\nabla_x u_n(t,\cdot),\nu\big\rangle\,d\mathcal{H}^{n-1}(x)\\
&+\varepsilon^p\int_{\partial B(0,n)}\big\langle\beta(t,\cdot),\nu\big\rangle\,d\mathcal{H}^{n-1}(x)+\varepsilon p\int_{B(0,n)}c(t,\cdot)\big(u_n^\varepsilon(t,\cdot)\big)^{p-1}\,dx,
\end{aligned}
$$

from (5.9) we get

$$
\frac{d}{dt}\big\|u_n^\varepsilon(t,\cdot)\big\|_{L^p(B(0,n))}^p\leqslant g_n^{\varepsilon,p}(t)+pK_p\big\|u_n^\varepsilon(t,\cdot)\big\|_{L^p(B(0,n))}^p,\quad t\in[s,b].
$$

Hence, we easily deduce that

$$
\big\|u_n^\varepsilon(t,\cdot)\big\|_{L^p(B(0,n))}^p\leqslant e^{pK_p(t-s)}\big\|u_n^\varepsilon(s,\cdot)\big\|_{L^p(B(0,n))}^p+\int_s^t e^{pK_p(t-\tau)}g_n^{\varepsilon,p}(\tau)\,d\tau,
$$

and, by dominated convergence, (5.7) follows at once.

To prove (5.7) for $p=1$ it suffices to write it for $p>1$ and, then, let $p\to1^+$ since

$$
\lim_{p\to1^+}\|\psi\|_{L^p(B(0,n))}=\|\psi\|_{L^1(B(0,n))},
$$

for any $\psi\in C(\overline{B(0,n)})$.

Now, let $v_n(t,x)=u_n(t,x)\chi_{B(0,n)}(x)$. Then, $\lim_{n\to+\infty}v_n(t,x)=u(t,x)$ for $(t,x)\in(s,+\infty)\times\mathbb{R}^d$ and

$$
\begin{aligned}
\big\|u(t,\cdot)\big\|_{L^p(\mathbb{R}^d)}^p&\leqslant\liminf_{n\to+\infty}\big\|v_n(t,\cdot)\big\|_{L^p(\mathbb{R}^d)}^p=\liminf_{n\to+\infty}\big\|u_n(t,\cdot)\big\|_{L^p(B(0,n))}^p\\
&\leqslant\liminf_{n\to+\infty}e^{pK_p(t-s)}\|f\|_{L^p(B(0,n))}^p=e^{pK_p(t-s)}\|f\|_{L^p(\mathbb{R}^d)}^p,
\end{aligned}
$$

for any $t \in [s,b]$. Therefore, $G(t,s) \in \mathcal{L}(L^p(\mathbb{R}^d))$ for $1 \leqslant p < +\infty$ and $t \in [s,T]$, and it satisfies (5.6). This completes the proof. $\square$

The condition assumed in Theorem 5.3 is a sort of compensation between the diffusion coefficients, the drift and the potential of the operator $\mathcal{A}$. Note that in the case when $c \equiv 0$ and q_{ij} $(i,j=1,\dots,d)$ are constant with respect to the spatial variables, such a condition reduces to the request that the spatial divergence of the drift b is bounded from below. Slightly modifying the proof of the previous theorem, we can give another sufficient condition for $L^p(\mathbb{R}^d)$ be preserved by the action of the evolution operator $(G(t,s))$, which applies to some situation where condition (5.4) is not satisfied (see Remark 6.7).

Theorem 5.4. *Fix $p>1$, $a,b \in I$ with $a<b$. Assume that the diffusion coefficients q_{ij} $(i,j=1,\dots,d)$ are continuously differentiable with respect to the spatial variables in $[a,b]\times\mathbb{R}^d$, the function $\eta(t,\cdot)$ in Hypothesis 2.1(i) is measurable for any $t\in[a,b]$ and*

$$\frac{|\beta(t,x)|^2}{4(p-1)\eta(t,x)} - c(t,x) \leqslant K'_p, \qquad (t,x)\in[a,b]\times\mathbb{R}^d \tag{5.10}$$

(see (5.5)) for some positive constant K'_p. Then, $L^p(\mathbb{R}^d)$ is invariant under $G(t,s)$ for any $(t,s)\in\Lambda_{[a,b]}$. Moreover,

$$\left\|G(t,s)f\right\|_{L^p(\mathbb{R}^d)} \leqslant e^{K'_p(t-s)}\|f\|_{L^p(\mathbb{R}^d)}, \qquad a\leqslant s\leqslant t\leqslant b. \tag{5.11}$$

Proof. The main difference with respect to the proof of Theorem 5.3 is in the estimate of the term

$$I := \int_{B(0,n)} \left(u_n^\varepsilon(t,\cdot)\right)^{p-1}\left\langle\beta(t,\cdot),\nabla_x u_n(t,\cdot)\right\rangle dx.$$

Using Hölder and Young's inequalities we can estimate

$$\begin{aligned}
I &\leqslant \int_{B(0,n)} \sqrt{\eta(t,\cdot)}\left(u_n^\varepsilon(t,\cdot)\right)^{\frac{p}{2}-1}\left|\nabla_x u_n(t,\cdot)\right|\frac{1}{\sqrt{\eta(t,\cdot)}}\left|\beta(t,\cdot)\right|\left(u_n^\varepsilon(t,\cdot)\right)^{\frac{p}{2}}dx \\
&\leqslant \left(\int_{B(0,n)} \eta(t,\cdot)\left(u_n^\varepsilon(t,\cdot)\right)^{p-2}\left|\nabla_x u_n(t,\cdot)\right|^2 dx\right)^{\frac12}\left(\int_{B(0,n)} \frac{|\beta(t,\cdot)|^2}{\eta(t,\cdot)}\left(u_n^\varepsilon(t,\cdot)\right)^p dx\right)^{\frac12} \\
&\leqslant \delta\int_{B(0,n)} \eta(t,\cdot)\left(u_n^\varepsilon(t,\cdot)\right)^{p-2}\left|\nabla_x u_n(t,\cdot)\right|^2 dx + \frac{1}{4\delta}\int_{B(0,n)} \frac{|\beta(t,\cdot)|^2}{\eta(t,\cdot)}\left(u_n^\varepsilon(t,\cdot)\right)^p dx,
\end{aligned}$$

for any $\delta>0$. Hence,

$$\begin{aligned}
\int_{B(0,n)} \left(u_n^\varepsilon(t,\cdot)\right)^{p-1}\mathcal{A}(t)u_n(t,\cdot)\,dx \leqslant{}& \varepsilon^{p-1}\int_{\partial B(0,n)} \left\langle Q(t,\cdot)\nabla_x u_n(t,\cdot),\nu\right\rangle d\mathcal{H}^{n-1}(x) \\
&+ \int_{B(0,n)} \left(\frac{|\beta(t,\cdot)|^2}{4\delta\eta(t,\cdot)} - c(t,\cdot)\right)\left(u_n^\varepsilon(t,\cdot)\right)^p dx \\
&- (p-1-\delta)\int_{B(0,n)} \eta(t,\cdot)\left(u_n^\varepsilon(t,\cdot)\right)^{p-2}\left|\nabla_x u_n(t,\cdot)\right|^2 dx \\
&+ \varepsilon\int_{B(0,n)} c(t,\cdot)\left(u_n^\varepsilon(t,\cdot)\right)^{p-1} dx.
\end{aligned}$$

The optimal choice $\delta = p-1$ gives

$$\begin{aligned}
\int_{B(0,n)} \left(u_n^\varepsilon(t,\cdot)\right)^{p-1}\mathcal{A}(t)u_n(t,\cdot)\,dx \leqslant{}& \varepsilon^{p-1}\int_{\partial B(0,n)} \left\langle Q(t,\cdot)\nabla_x u_n(t,\cdot),\nu\right\rangle d\mathcal{H}^{n-1}(x) \\
&+ K'_p\int_{B(0,n)} \left(u_n^\varepsilon(t,\cdot)\right)^p dx + \varepsilon\int_{B(0,n)} c(t,\cdot)\left(u_n^\varepsilon(t,\cdot)\right)^{p-1} dx.
\end{aligned}$$

Now, we can conclude arguing as in the proof of Theorem 5.3. $\square$

Remark 5.5. We stress that conditions (5.4) and (5.10) are not comparable to each other. The one-dimensional nonautonomous Ornstein–Uhlenbeck operator $\mathcal{A}(t)\psi = \psi'' + (t^2+1)x\psi'$ ($t \in \mathbb{R}$) satisfies condition (5.4) but it does not satisfy condition (5.10). On the other hand, as we show in Remark 6.7, there are situations where condition (5.10) holds and (5.4) is not satisfied.

To conclude this section, let us prove the following consequence of Theorems 5.3 and 5.4.

Corollary 5.6. *Let the assumptions of Theorem* 5.3 (*resp. Theorem* 5.4) *be satisfied. Then, $G(t,s)f$ tends to f in $L^p(\mathbb{R}^d)$ as $t \to s^+$, for any $f \in L^p(\mathbb{R}^d)$ and any $s \in [a,b)$. Moreover, for any $t \in (a,b]$, $G(t,\cdot)f \in C([a,t]; L^p(\mathbb{R}^d))$.*

Proof. It suffices to prove the first assertion in the case when $f \in C^2_c(\mathbb{R}^d)$. The general case then will follow by a standard density argument through estimate (5.6) (resp. (5.11)).

So, let us fix $f \in C^2_c(\mathbb{R}^d)$, $s,t \in I$ with $s < t$. Since $f = G(t,t)f$, from formula (3.5) it follows that

$$\big(G(t,s)f\big)(x) - f(x) = \int_s^t \big(G(t,r)\mathcal{A}(r)f\big)(x)\,dr, \quad x \in \mathbb{R}^d.$$

Hence, using estimate (5.6) (resp. (5.11)) we can infer that

$$\begin{aligned}\big\|G(t,s)f - f\big\|_{L^p(\mathbb{R}^d)} &\leqslant \sup_{r\in[a,b]}\big\|\mathcal{A}(r)f\big\|_{L^p(\mathbb{R}^d)}\int_s^t e^{M_p(t-r)}\,dr\\ &= \sup_{r\in[a,b]}\big\|\mathcal{A}(r)f\big\|_{L^p(\mathbb{R}^d)} M_p^{-1}\big(e^{M_p(t-s)}-1\big),\end{aligned} \tag{5.12}$$

where $M_p = K_p$ (resp. $M_p = K'_p$). Estimate (5.12) clearly shows that $G(t,s)f$ tends to f in $L^p(\mathbb{R}^d)$ as $t \to s^+$.

The last assertion of the corollary can be proved likewise. □

6. Examples

In this section we exhibit some classes of operators to which the main results of this paper apply.

6.1. A class of operators to which Theorem 4.4 applies

Let $\mathcal{A}$ be the differential operator defined by

$$\big(\mathcal{A}(t)\psi\big)(x) = \omega(t)\big(1+|x|^2\big)^k\Delta\psi(x) + \big\langle b(t,x), \nabla\psi(x)\big\rangle - c(t,x)\big(1+|x|^2\big)^m\psi(x), \tag{6.1}$$

for any $(t,x) \in I \times \mathbb{R}^d$, on smooth functions $\psi : \mathbb{R}^d \to \mathbb{R}$.

Hypothesis 6.1.

(i) $k, m \in \mathbb{N}$;
(ii) $\omega \in C^{\alpha/2}_{\rm loc}(I)$ satisfies $\inf_{t\in I}\omega(t) > 0$, $b \in C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d,\mathbb{R}^d)$ and $c \in C^{\alpha/2,\alpha}_{\rm loc}(I\times\mathbb{R}^d)$ is positive and bounded;
(iii) there exist $l \in \mathbb{N}$ such that $l > (m+2)\vee k$, $R > 0$ and a continuous function $C_1 : I \to (0,+\infty)$ such that

$$\big\langle b(t,x), x\big\rangle \leqslant -C_1(t)\big(1+|x|^2\big)^l, \quad (t,x) \in I \times \mathbb{R}^d \setminus B(0,R).$$

Under such assumptions the operator $G(t,s)$ associated with the operator $\mathcal{A}$ in (6.1) is compact in $C_b(\mathbb{R}^d)$ for any $t,s \in I$ with $s < t$. To check the claim it suffices to show that, for any bounded interval $J \subset I$, there exist a positive and bounded smooth function $W : \mathbb{R}^d \to \mathbb{R}$, with positive infimum, a positive smooth function $\zeta : \mathbb{R}^d \to \mathbb{R}$, blowing up as $|x| \to +\infty$, an increasing strictly convex function $h : [0,+\infty) \to \mathbb{R}$, with $1/h$ being integrable in a neighborhood of $+\infty$, and $\mu \in \mathbb{R}$, $R_0 > 0$ such that

$$\begin{aligned}&\text{(i)} \quad \big(\mathcal{A}(t)W\big)(x) - \mu W(x) \geqslant 0, \quad (t,x) \in J \times \mathbb{R}^d \setminus B(0,R_0),\\ &\text{(ii)} \quad \big(\mathcal{A}(t)\zeta\big)(x) \leqslant -h\big(\zeta(x)\big), \quad (t,x) \in J \times \mathbb{R}^d.\end{aligned} \tag{6.2}$$

We have also to show that Hypothesis 3.3 is fulfilled. For notational convenience we set $\omega_0 = \sup_{t\in J}\omega(t)$.

To prove the first condition in (6.2), we set $W(x) = 1 + \frac{1}{1+|x|^2}$ for any $x \in \mathbb{R}^d$. Then,

$$\begin{aligned}\big(\mathcal{A}(t)W\big)(x)-\mu W(x)&=-2\,d\omega(t)\,\big(1+|x|^2\big)^{k-2}+8\omega(t)|x|^2\big(1+|x|^2\big)^{k-3}\\&\quad-2\frac{\langle b(t,x),x\rangle}{(1+|x|^2)^2}-\big(c(t,x)\big(1+|x|^2\big)^m+\mu\big)\bigg(1+\frac{1}{1+|x|^2}\bigg)\\&\geqslant 2\big(1+|x|^2\big)^{l-2}\big\{C_1(t)-d\omega(t)\,\big(1+|x|^2\big)^{k-l}\\&\quad-c(t,x)\big(1+|x|^2\big)^{m-l+2}-|\mu|\big(1+|x|^2\big)^{2-l}\big\}\\&\geqslant 2\big(1+|x|^2\big)^{l-2}\Big\{C_1(t)-d\omega_0\,R_0^{2k-2l}-|\mu|R_0^{4-2l}-R_0^{2m-2l+4}\sup_{(t,x)\in J\times\mathbb{R}^d}c(t,x)\Big\},\end{aligned}$$

for any $(t,x)\in J\times\mathbb{R}^d\setminus B(0,R_0)$ and any $R_0>R$. Hence, condition (6.2)(i) follows for any $\mu\in\mathbb{R}$, provided we take R_0 sufficiently large.

Let us now check condition (6.2)(ii). For this purpose, we set $\zeta(x)=1+|x|^2$ for any $x\in\mathbb{R}^d$. Then,

$$\begin{aligned}\big(\mathcal{A}(t)\zeta\big)(x)&=2\,d\omega(t)\,\big(1+|x|^2\big)^k+2\big\langle b(t,x),x\big\rangle-c(t,x)\big(1+|x|^2\big)^{m+1}\\&\leqslant 2\,d\omega(t)\,\big(1+|x|^2\big)^k-2C_1(t)\big(1+|x|^2\big)^l\\&=\big(1+|x|^2\big)^l\big\{-2C_1(t)+2\,d\omega(t)\,\big(1+|x|^2\big)^{k-l}\big\}\\&\leqslant 2\big(1+|x|^2\big)^l\big\{-C_1(t)+\omega_0\,d\big(1+|x|^2\big)^{-1}\big\},\end{aligned}$$

where in the last inequality we have used the fact that $l\geqslant k+1$. We now observe that, for any $\varepsilon>0$, any $a>0$ and any $p\in\mathbb{N}$, $p\geqslant 2$, it holds that

$$ar\leqslant\varepsilon+\varepsilon^{1-p}p^{-p}(p-1)^{p-1}a^pr^p=:\varepsilon+C_\varepsilon a^pr^p,\quad r>0.$$

Applying this inequality with

$$r=\frac{1}{1+|x|^2},\qquad a=\omega_0 d,\qquad p=l>2,$$

we can estimate

$$\big(\mathcal{A}(t)\zeta\big)(x)\leqslant -2\big(C_1(t)-\varepsilon\big)\big(1+|x|^2\big)^l+2C_\varepsilon(\omega_0 d)^l,\quad (t,x)\in J\times\mathbb{R}^d.$$

Fix $2\varepsilon<\inf_{t\in J}C_1(t)=:\gamma$. With this choice of ε we get

$$\big(\mathcal{A}(t)\zeta\big)(x)\leqslant -\gamma\big(1+|x|^2\big)^l+2C_\varepsilon(\omega_0 d)^l=:-\gamma\big(1+|x|^2\big)^l+C'_\varepsilon.$$

Now, we introduce the function $h:[0,+\infty)\to\mathbb{R}$ defined by $h(t)=\gamma t^l-C'_\varepsilon$ for any $t\geqslant 0$. Clearly, h is strictly increasing, convex and $1/h$ is integrable in a neighborhood of $+\infty$. Moreover, $(\mathcal{A}(t)\zeta)(x)\leqslant -h(\zeta(x))$ for any $t\in J$ and any $x\in\mathbb{R}^d$, i.e., condition (6.2)(ii) holds true.

Note that, in fact, we have shown that

$$\big(\mathcal{A}_{-c}(t)\zeta\big)(x)\leqslant -\gamma\big(1+|x|^2\big)^l+C'_\varepsilon,\quad t\in J,\ x\in\mathbb{R}^d.$$

In particular, this implies that $(\mathcal{A}_{-c}(t)\zeta)(x)\leqslant C'_\varepsilon\zeta(x)$ for any $(t,x)\in J\times\mathbb{R}^d$, which clearly implies Hypothesis 3.3.

6.2. A class of operators to which the results of Section 5 apply

Let $\mathcal{A}$ be defined by

$$\big(\mathcal{A}(t)\psi\big)(x)=\big(1+|x|^2\big)^m\operatorname{Tr}\big(Q\,(t,x)D^2\psi(x)\big)+\big(1+|x|^2\big)^rb(t)\big\langle x,\nabla\psi(x)\big\rangle-c(t,x)\psi(x),\tag{6.3}$$

where m,r are nonnegative constants. We assume the following set of assumptions on the coefficients of the operator $\mathcal{A}$, on m and r.

Hypothesis 6.2.

(i) $b\in C^{\alpha/2}_{\mathrm{loc}}(I)$, $b(t)\leqslant 0$ for any $t\in I$;
(ii) $c\in C^{\alpha/2,\alpha}_{\mathrm{loc}}(I\times\mathbb{R}^d)$ and, for any bounded interval $J\subset I$, there exist $C_J\geqslant 0$ and $q=q_J\geqslant 0$ such that $c(t,x)\geqslant C_J(1+|x|^2)^q$ for any $(t,x)\in J\times\mathbb{R}^d$;

(iii) there exists a positive constant η_0 such that

$$\langle Q(t,x)\xi,\xi\rangle \geqslant \eta_0|\xi|^2, \quad \xi\in\mathbb{R}^d,\ (t,x)\in I\times\mathbb{R}^d.$$

Moreover,

$$M_J^{(1)} := \sup_{(t,x)\in J\times\mathbb{R}^d}\big|Q(t,x)\big|_{\mathbb{R}^{d^2}} < +\infty,$$

for any bounded interval $J\subset I$;

(iv) for any bounded interval $J\subset I$ one of the following conditions is satisfied:
(a) $r>m-1$ and $b(t)<0$ in J;
(b) $q_J>m-1$ and $C_J>0$;

(v) there exists a compact set $[a,b]\subset I$ such that $C_{[a,b]}>0$ and $q_{[a,b]}>\max\{r,m-1,1\}$.

Under the previous conditions, Hypothesis 2.1 is satisfied. Of course, we have to check only Hypothesis 2.1(iv). For this purpose we take $\varphi(x)=1+|x|^2$ for any $x\in\mathbb{R}^d$. As it is easily seen

$$\big(\mathcal{A}(t)\varphi\big)(x) = 2\,\mathrm{Tr}\big(Q(t,x)\big)\big(1+|x|^2\big)^m + 2b(t)|x|^2\big(1+|x|^2\big)^r - c(t,x)\big(1+|x|^2\big),$$

for any $(t,x)\in I\times\mathbb{R}^d$. Hence,

$$\big(\mathcal{A}(t)\varphi\big)(x) \leqslant 2\sqrt{d}M_J^{(1)}\big(1+|x|^2\big)^m + 2b(t)|x|^2\big(1+|x|^2\big)^r - C_J\big(1+|x|^2\big)^{q+1},$$

for any $(t,x)\in J\times\mathbb{R}^d$ and any bounded interval $J\subset I$. It is now easy to show that, under Hypothesis 6.2(iv-a) or 6.2(iv-b)

$$R_J := \sup_{(t,x)\in J\times\mathbb{R}^d}\big(\mathcal{A}(t)\varphi\big)(x) < +\infty.$$

Hence, Hypothesis 2.1(iv) is satisfied with $\lambda = R_J\vee 0$.

We now consider the function $V:\mathbb{R}^d\to\mathbb{R}$ defined by $V(x)=(1+|x|^2)^{-1}$ for any $x\in\mathbb{R}^d$. A straightforward computation shows that

$$\begin{aligned}\big(\mathcal{A}(t)V\big)(x) &= 8\langle Q(t,x)x,x\rangle\big(1+|x|^2\big)^{m-3} - 2\,\mathrm{Tr}\big(Q(t,x)\big)\big(1+|x|^2\big)^{m-2}\\ &\quad - 2b(t)|x|^2\big(1+|x|^2\big)^{r-2} - c(t,x)\big(1+|x|^2\big)^{-1},\end{aligned}$$

for any $(t,x)\in I\times\mathbb{R}^d$. Hence,

$$\big(\mathcal{A}(t)V\big)(x) \leqslant 8M_{[a,b]}^{(1)}|x|^2\big(1+|x|^2\big)^{m-3} + 2\|b\|_{L^\infty((a,b))}|x|^2\big(1+|x|^2\big)^{r-2} - C_{[a,b]}\big(1+|x|^2\big)^{q-1},$$

for any $(t,x)\in[a,b]\times\mathbb{R}^d$. Therefore, taking Hypothesis 6.2(v) into account, we can conclude that

$$\lim_{|x|\to+\infty}\sup_{t\in[a,b]}\big(\mathcal{A}(t)V\big)(x) = -\infty.$$

In particular, there exists $R>0$ such that $(\mathcal{A}(t)V)(x)\leqslant 0$ for any $(t,x)\in[a,b]\times\mathbb{R}^d\setminus\overline{B(0,R)}$. Therefore, the condition

$$\big(\mathcal{A}(t)V\big)(x) \leqslant \lambda_0 V(x), \quad (t,x)\in[a,b]\times\mathbb{R}^d,$$

is satisfied with $\lambda_0=(1+R^2)(\sup_{(t,x)\in[a,b]\times B(0,R)}(\mathcal{A}(t)V)(x))^+$. Here, $(\cdot)^+$ denotes the positive part of the quantity in brackets. As a byproduct, we get the following:

Proposition 6.3. *Under Hypothesis* 6.2 *the evolution operator* $(G(t,s))$ *associated with the operator* $\mathcal{A}$ *in* (6.3) *preserves* $C_0(\mathbb{R}^d)$ *for any* $s,t\in\Lambda_{[a,b]}$.

Let us now compute the divergence of the vector field β defined in (5.5) for the operator $\mathcal{A}$ in (6.3). For this purpose, we assume the following additional assumptions on the coefficients of the operator $\mathcal{A}$, on m, q and r.

Hypothesis 6.4.

(i) The diffusion coefficients q_{ij} $(i,j=1,\dots,d)$ are continuously differentiable in $I\times\mathbb{R}^d$ with respect to the spatial variables and $\nabla_x q_{ij}$ is bounded in $[a,b]\times\mathbb{R}^d$ (where $[a,b]$ is as in Hypothesis 6.2(v)) for any $i,j=1,\dots,d$. Moreover, the second-order weak spatial derivatives $D_{ij}q_{ij}$ $(i,j=1,\dots,d)$ exist and are bounded functions in $[a,b]\times\mathbb{R}^d$;

(ii) $q_{[a,b]}>\max\{r,m,1\}$.

Under such additional assumptions we get

$$\begin{aligned}\mathrm{div}_x\,\beta(t,x)+c(t,x) &= -4m\big(1+|x|^2\big)^{m-1}\sum_{i,j=1}^{d} D_i q_{ij}(t,x)x_j-\big(1+|x|^2\big)^m\sum_{i,j=1}^{d} D_{ij}q_{ij}(t,x)\\ &\quad -4m(m-1)\big(1+|x|^2\big)^{m-2}\big\langle Q(t,x)x,x\big\rangle-2m\big(1+|x|^2\big)^{m-1}\mathrm{Tr}\big(Q(t,x)\big)\\ &\quad +b(t)\big(1+|x|^2\big)^{r-1}\big(d+(2r+d)|x|^2\big)+c(t,x),\end{aligned}$$

for any $(t,x)\in I\times\mathbb{R}^d$. Hence, we can estimate

$$\begin{aligned}\mathrm{div}_x\,\beta(t,x)+c(t,x) &\geqslant -4mM^{(2)}_{[a,b]}|x|\big(1+|x|^2\big)^{m-1}-\big(1+|x|^2\big)^m M^{(3)}_{[a,b]}\\ &\quad -4m(m-1)M^{(1)}_{[a,b]}\big(1+|x|^2\big)^{m-1}-2m\sqrt{d}M^{(1)}_{[a,b]}\big(1+|x|^2\big)^{m-1}\\ &\quad -\|b\|_{L^\infty((a,b))}\big(1+|x|^2\big)^{r-1}\big(d+(2r+d)|x|^2\big)+C_{[a,b]}\big(1+|x|^2\big)^q,\end{aligned}$$

for any $(t,x)\in[a,b]\times\mathbb{R}^d$, where

$$M^{(2)}_{[a,b]}=\sup_{(t,x)\in[a,b]\times\mathbb{R}^d}\left(\sum_{j=1}^{d}\left(\sum_{i=1}^{d}\big|D_i q_{ij}(t,x)\big|\right)^2\right)^{\frac{1}{2}},\qquad M^{(3)}_{[a,b]}=\sum_{i,j=1}^{d}\sup_{(t,x)\in[a,b]\times\mathbb{R}^d}\big|D_{ij}q_{ij}(t,x)\big|.$$

Due to the conditions imposed on m,q,r, $\mathrm{div}_x\,\beta(t,x)+c(t,x)$ tends to $+\infty$ as $|x|\to+\infty$, uniformly with respect to $t\in[a,b]$. We have so proved the following:

Proposition 6.5. *Under Hypotheses* 6.2 *and* 6.4 *the evolution operator* $(G(t,s))$ *associated with the operator* $\mathcal{A}$ *in* (6.3) *preserves* $L^p(\mathbb{R}^d)$ *for any* $p\in[1,+\infty)$ *and any* $s,t\in\Lambda_{[a,b]}$.

Finally, observe that, arguing as above, one can easily verify that, if

$$q_{[a,b]}>\max\{m,2r+1-m\},\tag{6.4}$$

then the condition (5.10) is fulfilled. Hence,

Proposition 6.6. *Let Hypotheses* 6.2(i)–(iv) *and condition* (6.4) *be fulfilled. Further, assume that the diffusion coefficients* q_{ij} $(i,j=1,\dots,d)$ *are continuously differentiable with respect to the spatial variables and assume that* $C_{[a,b]}>0$. *Then, the evolution operator* $(G(t,s))$ *associated with the operator* $\mathcal{A}$ *in* (6.3) *preserves* $L^p(\mathbb{R}^d)$ *for any* $s,t\in\Lambda_{[a,b]}$.

Remark 6.7. In this example, condition (6.4) trivially implies Hypothesis 6.4(ii). Hence, the difference between Propositions 6.5 and 6.6 is just in the smoothness of the coefficients. In general, as claimed before Theorem 5.4, even for smooth coefficients, condition (5.10) may hold also in some situations where condition (5.4) is not satisfied. Consider for instance the operator $\mathcal{A}$ defined by

$$\big(\mathcal{A}(t)\varphi\big)(x)=\Delta\varphi(x)-\frac{t^2+2}{t^2+1}\big(2+\sin\big(|x|^4\big)\big)\big\langle x,\nabla\varphi(x)\big\rangle-\big(t^2+1\big)\big(1+|x|^2\big)^q\varphi(x),$$

for any $(t,x)\in\mathbb{R}^{1+d}$, on smooth functions φ.

A straightforward computation shows that operator $\mathcal{A}$ satisfies Hypothesis 2.1. On the other hand, condition (5.4) is satisfied, by any $[a,b]\subset I$ provided that $q_{[a,b]}>2$, whereas condition (6.4) is satisfied (by any $p>1$ and any $[a,b]$ as above) provided that $q_{[a,b]}>1$.

Acknowledgments

The authors wish to thank A. Lunardi, G. Metafune and D. Pallara, for fruitful discussions, and the anonymous referee for the careful reading of the manuscript and some useful comments.

References

[1] P. Acquistapace, Evolution operators and strong solutions of abstract linear parabolic equations, Differential Integral Equations 1 (1988) 433–457.
[2] L. Angiuli, G. Metafune, C. Spina, Feller semigroups and invariant measures, Riv. Mat. Univ. Parma 1 (2010) 347–406.
[3] R. Azencott, Behaviour of diffusion semigroups at infinity, Bull. Soc. Math. France 102 (1974) 193–240.
[4] M. Bertoldi, L. Lorenzi, Analytical Methods for Markov Semigroups, Chapman & Hall/CRC, Boca Raton, 2007.
[5] G. Da Prato, A. Lunardi, Ornstein–Uhlenbeck operators with time periodic coefficients, J. Evol. Equ. 7 (2007) 587–614.

[6] G. Da Prato, M. Röckner, A note on evolution systems of measures for time-dependent stochastic differential equations, in: Seminar on Stochastic Analysis, Random Fields and Applications V, Progr. Probab., vol. 59, Birkhäuser, Basel, 2008, pp. 115–122.
[7] E.B. Dynkin, Three classes of infinite-dimensional diffusions, J. Funct. Anal. 86 (1989) 75–110.
[8] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall, Englewood Cliffs, NJ, 1964.
[9] M. Geissert, A. Lunardi, Invariant measures and maximal L^2 regularity for nonautonomous Ornstein–Uhlenbeck equations, J. London Math. Soc. (2) 77 (2008) 719–740.
[10] M. Geissert, A. Lunardi, Asymptotic behavior and hypercontractivity in nonautonomous Ornstein–Uhlenbeck equations, J. London Math. Soc. (2) 79 (2009) 85–106.
[11] S. Itô, Fundamental solutions of parabolic differential equations and boundary value problems, Jpn. J. Math. 27 (1957) 55–102.
[12] M. Kunze, L. Lorenzi, A. Lunardi, Nonautonomous Kolmogorov parabolic equations with unbounded coefficients, Trans. Amer. Math. Soc. 362 (2010) 169–198.
[13] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, Nauka, Moscow, 1967, English transl.: American Mathematical Society, Providence, RI, 1968.
[14] A. Lunardi, Compactness and asymptotic behavior in nonautonomous linear parabolic equations with unbounded coefficients in $\mathbb{R}^d$, in: Parabolic Problems. The Herbert Amann Festschrift, Progr. Nonlinear Differential Equations Appl., vol. 80, 2010.
[15] G. Metafune, D. Pallara, M. Wacker, Feller semigroups on $\mathbb{R}^N$, Semigroup Forum 65 (2002) 159–205.
[16] G. Metafune, D. Pallara, M. Wacker, Compactness properties of Feller semigroups, Studia Math. 153 (2002) 179–206.
[17] K.T. Sturm, Harnack's inequality for parabolic operators with singular low order terms, Math. Z. 216 (1994) 593–612.